ON THE MARCINKIEWICZ–FEJÉR MEANS OF DOUBLE WALSH–KACZ–MARZ–FOURIER SERIES

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Abstract: In this paper we prove that the maximal operator of the Marcinkiewicz–Fejér means of the 2-dimensional Fourier series with respect to the Walsh–Kaczmarz system is not bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$.

The second author [5] proved that the maximal function of Marcinkiewicz–Fejér means with respect to the two dimensional Walsh–Kaczmarz system is of weak type $(1, 1)$ and of type $(p, p)$ for all $p > 1$. Consequently, for any integrable function $f$ the Marcinkiewicz–Fejér means with respect to the two dimensional Walsh–Kaczmarz system converge almost everywhere to the function itself. This theorem was extended in [2] by the authors and G. Gát. Namely, for $p > 2/3$, the maximal oper-

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ator $M^*$ is bounded from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$. The main aim of this paper is to prove that the assumption $p > 2/3$ is essential. Namely, the maximal operator $M^*$ is not bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$.

Let $\mathbf{P}$ denote the set of positive integers, $\mathbb{N} := \mathbf{P} \cup \{0\}$. Denote $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given such that the measure of a singleton is 1/2. Let $G$ be the complete direct product of the countable infinite copies of the compact groups $\mathbb{Z}_2$. The elements of $G$ are of the form $x = (x_0, x_1, \ldots, x_k, \ldots)$ with $x_k \in \{0, 1\}$ ($k \in \mathbb{N}$). The group operation on $G$ is the coordinate-wise addition, the measure (denote by $\mu$) and the topology are the product measure and topology. The compact Abelian group $G$ is called the Walsh group. A base for the neighborhoods of $G$ can be given in the following way:

$I_0 (x) := G, I_n (x) := I_n (x_0, \ldots, x_{n-1}) := \{y \in G : y = (x_0, \ldots, x_{n-1}, y_n, y_{n+1}, \ldots)\}$, $(x \in G, n \in \mathbb{N})$.

These sets are called dyadic intervals. Let $0 = (0 : i \in \mathbb{N}) \in G$ denote the null element of $G$, $I_n := I_n (0) \ (n \in \mathbb{N})$. Set $e_n := (0, \ldots, 0, 1, 0, \ldots) \in G$, the $n$th coordinate of which is 1 and the rest are zeros ($n \in \mathbb{N}$).

For $k \in \mathbb{N}$ and $x \in G$ denote

$\ r_k (x) := (-1)^{x_k}$

the $k$th Rademacher function. If $n \in \mathbb{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}$ ($i \in \mathbb{N}$), i.e. $n$ is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, that is $2^{[n]} \leq n < 2^{|n|+1}$.

The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$w_n (x) := \prod_{k=0}^{\infty} (r_k (x))^{n_k} = r_{|n|} (x) (-1)^{\sum_{k=0}^{[n]-1} n_k x_k} \quad (x \in G, n \in \mathbf{P}).$

The Walsh–Kaczmazr functions are defined by $\kappa_0 := 1$ and for $n \geq 1$

$\kappa_n (x) := r_{|n|} (x) \prod_{k=0}^{[n]-1} (r_{|n|-1-k} (x))^{n_k} = r_{|n|} (x) (-1)^{\sum_{k=0}^{[n]-1} n_k x_{[n]-k-1}}.$
For $A \in \mathbb{N}$ define the transformation $\tau_A : G \to G$ by

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_0, x_A, x_{A+1}, \ldots).$$

By the definition of $\tau_A$ (see [9]), we have

$$\kappa_n(x) = r_{|n|}(x)w_{n-2|n|}(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, \ x \in G).$$

The $\sigma$-algebra generated by the dyadic 2-dimensional cube $I_k^2$ of measure $2^{-k} \times 2^{-k}$ will be denoted by $F_k$ ($k \in \mathbb{N}$).

The Hardy martingale space $H_p(G^2)$ consists all martingales for which

$$\|f\|_{H_p} = \|f^*\|_p < \infty.$$
The Marcinkiewicz–Fejér means of a martingale \( f \) are defined by

\[
\mathcal{M}_n^\alpha (f; x^1, x^2) := \frac{1}{n} \sum_{k=0}^{n-1} S_{k,k}^\alpha (f, x^1, x^2).
\]

For the martingale \( f \) we consider the maximal operators

\[
\mathcal{M}^{*\kappa} f(x^1, x^2) = \sup_n |\mathcal{M}_n^\kappa(f, x^1, x^2)|.
\]

In 1939 for the two-dimensional trigonometric Fourier partial sums \( S_{j,j} (f) \) Marcinkiewicz [6] has proved for \( f \in L \log L([0, 2\pi]^2) \) that the means

\[
\mathcal{M}_n f = \frac{1}{n} \sum_{j=1}^{n} S_{j,j} (f)
\]

converge a.e. to \( f \) as \( n \to \infty \). Zhizhiashvili [14] improved this result for \( f \in L([0, 2\pi]^2) \).

For the two-dimensional Walsh–Fourier series Weisz [11] proved that the maximal operator

\[
\mathcal{M}^{*w} f = \sup_{n \geq 1} \frac{1}{n} \left| \sum_{j=0}^{n-1} S_{j,j}^w (f) \right|
\]

is bounded from the two-dimensional dyadic martingale Hardy space \( H_p \) to the space \( L_p \) for \( p > 2/3 \) and is of weak type \( (1,1) \). The first author [3] proved that the assumption \( p > 2/3 \) is essential for the boundedness of the maximal operator \( \mathcal{M}^{*w} \) from the Hardy space \( H_p(G^2) \) to the space \( L_p(G^2) \).

In 1974 Schipp [7] and Young [10] proved that the Walsh–Kaczmarz system is a convergence system. Gát [1] proved, for any integrable functions, that the Fejér means with respect to the Walsh–Kaczmarz system converge almost everywhere to the function itself. Gát’s Theorem was extended by Simon [8] to \( H_p \) spaces, namely that the maximal operator of Fejér means of one-dimensional Fourier series is bounded from Hardy
space $H_p(G^2)$ into the space $L_p(G^2)$ for $p > 1/2$.

The second author [5] proved, that for any integrable functions, the Marcinkiewicz–Fejér means with respect to the two dimensional Walsh–Kaczmarz system converge almost everywhere to the function itself. This theorem was extended in [2]. Namely, the following is true:

**Theorem A1.** Let $p > 2/3$, then the maximal operator $\mathcal{M}^{*\kappa}$ of the Marcinkiewicz–Fejér means of double Walsh–Kaczmarz–Fourier series is bounded from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$.

The aim of this paper is to prove that the assumption $p > 2/3$ is essential for the boundedness of the maximal operator $\mathcal{M}^{*\kappa}$ from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$. Namely, the following theorem holds:

**Theorem 1.** The maximal operator $\mathcal{M}^{*\kappa}$ of the Marcinkiewicz–Fejér means of double Walsh–Kaczmarz–Fourier series is not bounded from the Hardy space $H_{2/3}(G^2)$ to the space $L_{2/3}(G^2)$.

**Proof.** Let 

$$f_A(x^1, x^2) := (D_{2A+1}(x^1) - D_{2A}(x^1))(D_{2A+1}(x^2) - D_{2A}(x^2)).$$

It is simple to calculate

$$\hat{f}_{A}^\kappa(i, k) = \begin{cases} 1, & \text{if } i, k = 2A, \ldots, 2^{A+1} - 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$S_{i,j}^{\kappa}(f; x^1, x^2) = \begin{cases} (D_i^\kappa(x^1) - D_{2A}(x^1))(D_j^\kappa(x^2) - D_{2A}(x^2)), & \text{if } i, j = 2A + 1, \ldots, 2^{A+1} - 1, \\ f_A(x^1, x^2), & \text{if } i, j \geq 2^{A+1}, \\ 0, & \text{otherwise.} \end{cases}$$

We can write the $n$th Dirichlet kernel with respect to the Walsh–Kaczmarz system in the following form:

$$D_n^\kappa(x) = D_{2|n|}(x) + \sum_{k=2|n|}^{n-1} r_{|k|}(x)w_{k-2|n|}(\tau_{|k|}(x)) =$$

$$= D_{2|n|}(x) + r_{|n|}(x)D_{n-2|n|}^{\omega}(\tau_{|n|}(x)).$$

Thus, we have

$$\mathcal{M}^{\kappa}f_A(x^1, x^2) =$$

$$= \sup_{n \in \mathbb{N}} |\mathcal{M}_n^\kappa(f_A; x^1, x^2)| \geq \max_{1 \leq N \leq 2^A} |\mathcal{M}_{2A+N}^\kappa(f_A; x^1, x^2)| =$$
we decompose the set \( G \) to investigate the integral \( \int \). Goginava and K. Nagy have \( A > t \) where

\[
\text{We obtain } \| f_A \|_p = \| f_A^* \|_p = \| D_{2^A} \|_p^2 = 2^{2A(1-1/p)}.
\]

We obtain

\[
\frac{\| M^{\kappa} f_A \|_{2^A}}{\| f_A \|_{2^A}} \geq \frac{1}{2A+1-2^{-A}} \left( \int \max_{1 \leq N \leq 2^A} (N|\mathcal{K}_N^w(\tau_A(x^1), \tau_A(x^2))|)^{2/3} d\mu(x^1, x^2) \right)^{3/2}.
\]

To investigate the integral \( \int_{G^2} \max_{1 \leq N \leq 2^A} (N|\mathcal{K}_N^w(\tau_A(x^1), \tau_A(x^2))|)^{2/3} d\mu(x^1, x^2) \), we decompose the set \( G \) as the following disjoint union

\[
G = I_A \cup \bigcup_{t=0}^{A-1} J_t^A,
\]

where \( A > t \geq 1 \) and \( J_t^A := \{ x \in G : x_{A-1} = \cdots = x_{A-t} = 0, x_{A-t-1} = 1 \} \), \( J_0^A := \{ x \in G : x_{A-1} = 1 \} \). Notice that, by the definition of \( \tau_A \) we have \( \tau_A(J_t^A) = I_t \setminus I_{t+1} \). By Cor. 2.4 in [4], for \((x^1, x^2) \in I_A \times I_A \)

\[
\mathcal{K}_2^w(x^1, x^2) = \frac{(2^A + 1) (2^A + 1)}{6}.
\]
Therefore,
\[
\int_{G \times G} \max_{1 \leq N \leq 2^A} \left( N \left| \mathcal{K}_N^w(\tau_A(x^1), \tau_A(x^2)) \right| \right)^{2/3} d\mu(x^1, x^2) \geq \\
\geq \sum_{t=1}^{A-1} \int_{J_t^A \times J_t^A} \max_{1 \leq N \leq 2^A} \left( N \left| \mathcal{K}_N^w(\tau_A(x^1), \tau_A(x^2)) \right| \right)^{2/3} d\mu(x^1, x^2) \geq \\
\geq \sum_{t=1}^{A-1} \int_{(I_t \setminus I_{t+1}) \times (I_t \setminus I_{t+1})} \left( 2^t \left| \mathcal{K}_{2^t}^w(\tau_A(x^1), \tau_A(x^2)) \right| \right)^{2/3} d\mu(x^1, x^2) = \\
= \sum_{t=1}^{A-1} \int_{(I_t \setminus I_{t+1}) \times (I_t \setminus I_{t+1})} \left( 2^t \left( \frac{2^t + 1)(2^{t+1} + 1)}{6} \right) \right)^{2/3} d\mu(x^1, x^2) \geq \\
\geq \sum_{t=1}^{A-1} \int_{(I_t \setminus I_{t+1}) \times (I_t \setminus I_{t+1})} \left( \frac{2^{3t}}{6} \right)^{2/3} d\mu(x^1, x^2) \geq \\
\geq c(A - 1).
\]
This completes the proof of the main theorem. ♦

References


