

THE UNITARY ANALOGUE OF PILLAI'S ARITHMETICAL FUNCTION II.

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Abstract. Let k be a positive integer and let $(a, b)_{*,k}$ denote the greatest k -th power divisor of a which is a unitary divisor of b . We introduce the function

$$P_k^*(n) = \sum_{i=1}^{n^k} (i, n^k)_{*,k}$$

and obtain the arithmetical evaluation of it and an asymptotic formula for the summatory function of P_k^* , which improves for $k = 1$ an earlier result of the author.

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1. INTRODUCTION

The arithmetical function

$$P(n) = \sum_{i=1}^n (i, n),$$

where (i, n) denotes the g.c.d. of i and n , was introduced by S. S. PILLAI [6]. It is easy to show that

$$(1.1) \quad P(n) = \sum_{de=n} d\phi(e),$$

where ϕ is Euler's totient function and formula (1.1) furnishes the asymptotic estimate (see [8])

$$(1.2) \quad \sum_{n \leq x} P(n) = \frac{3}{\pi^2} x^2 \log x + O(x^2).$$

Let k be a positive integer and let $(a, b)_k$ denote the largest common k -th power divisor of a and b . H. G. KOPETZKY [4] investigated the following generalization of Pillai's function

$$P_k(n) = \sum_{i=1}^{n^k} (i, n^k)_k = \sum_{1 \leq i_1, \dots, i_k \leq n} (i_1, \dots, i_k, n)^k,$$

where (i_1, \dots, i_k, n) stands for the g.c.d. of i_1, \dots, i_k and n . He deduced an asymptotic formula for the summatory function of P_k , which was improved by J. CHIDAMBARASWAMY and R. SITARAMACHANDRARAO [1] showing that

$$(1.3) \quad \sum_{n \leq x} P_k(n) = \frac{x^{k+1}}{(k+1)\zeta(k+1)} \left(\log x + 2C - \frac{1}{k+1} - \frac{\zeta'(k+1)}{\zeta(k+1)} \right) + O(x^{k+a+\varepsilon}),$$

for every $\varepsilon > 0$, where ζ denotes the Riemann zeta function, C is the Euler constant and a is the exponent appearing in Dirichlet's divisor problem ($1/4 \leq a < 1/3$), i.e. for every $\varepsilon > 0$

$$\sum_{n \leq x} \tau(n) = x \log x + (2C - 1)x + O(x^{a+\varepsilon}),$$

where $\tau(n)$ is the number of divisors of n .

In the proof of formula (1.3) the authors of paper [1] used the identity

$$P_k(n) = \sum_{de=n} \mu(d) e^k \tau(e),$$

where μ is the Möbius function. For $k = 1$ formula (1.3) gives a considerably sharper estimate for the summatory function of Pillai's arithmetical function than (1.2).

The author of the present paper introduced in [9] the unitary analogue of Pillai's function by

$$P^*(n) = \sum_{i=1}^n (i, n)_*,$$

where $(i, n)_*$ is the greatest divisor of i which is a unitary divisor of n . We recall that d is a unitary divisor of n (notation $d||n$) if $d|n$ and $(d, n/d) = 1$. Using that

$$P^*(n) = \sum_{d||n} d \phi^*(n/d),$$

where ϕ^* is the unitary analogue of Euler's function, see E. COHEN [2], we deduced the asymptotic estimate

$$(1.4) \quad \sum_{n \leq x} P^*(n) = \frac{3\alpha}{\pi^2} x^2 \log x + O(x^2),$$

where

$$(1.5) \quad \alpha = \prod_p \left(1 - \frac{1}{(p+1)^2} \right),$$

the product being extended over all primes p .

For a fixed positive integer k let $(a, b)_{*,k}$ denote the greatest k -th power divisor of a which is a unitary divisor of b . In this paper we define the function

$$P_k^*(n) = \sum_{i=1}^{n^k} (i, n^k)_{*,k},$$

representing the unitary analogue of the function P_k and obtain the arithmetical evaluation of it and an asymptotic formula for the summatory function of P_k^* , which is analogous to (1.3) and is sharper for $k = 1$ than the estimate (1.4). Our method is elementary and is based on the convolutional identity (2.3) and on formula (3.1).

2. ARITHMETICAL PROPERTIES

Let μ^* denote the unitary analogue of the Möbius function given by $\mu^*(n) = (-1)^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of n and let $\phi_k^*(n)$ denote the number of integers i in a complete residue system mod n^k such that $(i, n^k)_{*,k} = 1$. It is known, see K. NAGESWARA RAO [5], that

$$(2.1) \quad \phi_k^*(n) = \sum_{d|n} d^k \mu^*(n/d).$$

Proposition 1.

$$(2.2) \quad P_k^*(n) = \sum_{d|n} d^k \phi_k^*(n/d).$$

Proof. Group the numbers $i \in \{1, \dots, n^k\}$ according to the value $(i, n^k)_{*,k} = d^k$, where $d^k | i, d | n$ and $(i/d^k, n^k/d^k)_{*,k} = 1$. Hence $i = jd^k, (j, (n/d)^k)_{*,k} = 1, 1 \leq j \leq (n/d)^k$ and we obtain

$$P_k^*(n) = \sum_{d|n} d^k \sum_{\substack{1 \leq j \leq (n/d)^k \\ (j, (n/d)^k)_{*,k} = 1}} 1 = \sum_{d|n} d^k \phi_k^*(n/d).$$

Proposition 2.

$$(2.3) \quad P_k^*(n) = \sum_{d|n} d^k \tau^*(d) \mu^*(n/d),$$

where $\tau^*(n)$ denotes the number of the unitary divisors of n .

Proof. Let \times denote the unitary convolution, see E. COHEN [2], and let $E_k(n) = n^k$, for every $n \geq 1$. Then using (2.2) and (2.1) we get

$$P_k^* = E_k \times \phi_k^* = E_k \times E_k \times \mu^* = E_k \tau^* \times \mu^*.$$

Corollary 1. *The function P_k^* is multiplicative and if $n = p_1^{a_1} \dots p_r^{a_r}$, then $P_k^*(n) = (2p_1^{ka_1} - 1) \dots (2p_r^{ka_r} - 1)$.*

Proof. Using identity (2.2) the function P_k^* is the unitary convolution of two multiplicative functions, hence it is multiplicative.

Proposition 3. *Another representation of the function P_k^* is given by*

$$(2.4) \quad P_k^*(n) = \sum_{1 \leq i_1, \dots, i_k \leq n} ((i_1, \dots, i_k), n)_*^k,$$

Proof. Group the vectors (i_1, \dots, i_k) with $1 \leq i_1, \dots, i_k \leq n$ according to the value $((i_1, \dots, i_k), n)_* = d$, where $d | (i_1, \dots, i_k), d | n$ and $((i_1, \dots, i_k)/d, n/d)_* = 1$. Hence $i_1 = j_1 d, \dots, i_k = j_k d, ((j_1, \dots, j_k), n/d)_* = 1, 1 \leq j_1, \dots, j_k \leq n/d$ and we get

$$\sum_{1 \leq i_1, \dots, i_k \leq n} ((i_1, \dots, i_k), n)_*^k = \sum_{d|n} d^k \sum_{\substack{1 \leq j_1, \dots, j_k \leq n/d \\ ((j_1, \dots, j_k), n/d)_* = 1}} 1 = \sum_{d|n} d^k J_k^*(n/d),$$

where J_k^* is the unitary analogue of the Jordan totient function and $J_k^* = \phi_k^*$, see [5], which completes the proof.

3. ASYMPTOTIC FORMULAE

We need the following results.

Proposition 4. (M. V. SUBBARAO and D. SURYANARAYANA [7]) *If u is a positive integer, then*

$$(3.1) \quad \sum_{\substack{n \leq x \\ (n,u)=1}} \tau^*(n) = \frac{\phi(u)x}{\zeta(2)\psi(u)} \left(\log x + 2C - 1 + 2\alpha(u) - 2\beta(u) - 2\frac{\zeta'(2)}{\zeta(2)} \right) \\ + O(S(u)\sqrt{x} \log x),$$

where the O -estimate is uniform in x and u ,

$$\psi(u) = u \prod_{p|u} \left(1 + \frac{1}{p}\right)$$

stands for the Dedekind function and

$$(3.2) \quad \alpha(u) = \sum_{p|u} \frac{\log p}{p-1}, \quad \beta(u) = \sum_{p|u} \frac{\log p}{p^2-1}, \quad S(u) = \sum_{d|u} \frac{3^{\omega(d)}}{\sqrt{d}}.$$

Remark. According to [3, p.154] and [7, p.5] we have

$$(3.3) \quad \alpha(u) = O(\log \log 3u) = O(\log u), \quad \beta(u) = O(1).$$

By partial summation we immediately have

Proposition 5. *If k and u are positive integers, then*

$$(3.4) \quad \sum_{\substack{n \leq x \\ (n,u)=1}} n^k \tau^*(n) = \frac{\phi(u)x^{k+1}}{(k+1)\zeta(2)\psi(u)} \left(\log x + 2C - \frac{1}{k+1} + 2\alpha(u) - 2\beta(u) - 2\frac{\zeta'(2)}{\zeta(2)} \right) \\ + O(S(u)x^k \sqrt{x} \log x),$$

where the O -estimate is uniform in x and u .

Proposition 6. *If k is a positive integer, then the series*

$$\sum_{n=1}^{\infty} \frac{\mu^*(n)\phi(n)}{n^{k+1}\psi(n)}$$

is absolutely convergent and its sum is given by

$$(3.5) \quad \alpha_k = \prod_p \left(1 - \frac{p-1}{(p+1)(p^{k+1}-1)} \right),$$

the product being extended over all primes p .

Proof. The absolute convergence follows at once by

$$\left| \frac{\mu^*(n)\phi(n)}{n^{k+1}\psi(n)} \right| \leq \frac{1}{n^{k+1}} \leq \frac{1}{n^2}.$$

The general term is a multiplicative function in n and expanding the series into an infinite product of Euler type we obtain (3.5).

Proposition 7. *If k is a positive integer, then*

$$(3.6) \quad \sum_{n \leq x} \frac{S(n)}{n^k \sqrt{n}} = O(1),$$

where $S(n)$ is defined by (3.2).

Proof.

$$\begin{aligned} \sum_{n \leq x} \frac{S(n)}{n^k \sqrt{n}} &= \sum_{de=n \leq x} \frac{3^{\omega(e)}}{d^k \sqrt{d} e^{k+1}} = \sum_{d \leq x} \frac{1}{d^k \sqrt{d}} \sum_{e \leq x/d} \frac{3^{\omega(e)}}{e^{k+1}} \\ &= \sum_{d \leq x} \frac{1}{d^k \sqrt{d}} \cdot O(1) = O\left(\sum_{d \leq x} \frac{1}{d^k \sqrt{d}}\right) = O(1), \end{aligned}$$

since $k \geq 1$ and $3^{\omega(n)} = O(n^\varepsilon)$, $\varepsilon > 0$.

Our main result is the following

Theorem. *If k is a positive integer, then*

$$(3.7) \quad \sum_{n \leq x} P_k^*(n) = \frac{x^{k+1}}{(k+1)\zeta(2)} \left(\alpha_k \left(\log x + 2C - \frac{1}{k+1} - 2\frac{\zeta'(2)}{\zeta(2)} \right) - A_k + 2B_k - 2C_k \right) \\ + O(x^k \sqrt{x} \log x),$$

where α_k is given by (3.5) and

$$A_k = \sum_{n=1}^{\infty} \frac{\mu^*(n)\phi(n) \log n}{n^{k+1}\psi(n)}, \\ B_k = \sum_{n=1}^{\infty} \frac{\mu^*(n)\phi(n)\alpha(n)}{n^{k+1}\psi(n)}, \quad C_k = \sum_{n=1}^{\infty} \frac{\mu^*(n)\phi(n)\beta(n)}{n^{k+1}\psi(n)}.$$

Proof. By (2.3), (3.4) and (3.3) we get

$$\begin{aligned} \sum_{n \leq x} P_k^*(n) &= \sum_{\substack{de=n \leq x \\ (d,e)=1}} \mu^*(d) e^k \tau^*(e) = \sum_{d \leq x} \mu^*(d) \sum_{\substack{e \leq x/d \\ (e,d)=1}} e^k \tau^*(e) \\ &= \sum_{d \leq x} \mu^*(d) \left(\frac{\phi(d)x^{k+1}}{(k+1)\zeta(2)\psi(d)d^{k+1}} \left(\log \frac{x}{d} + 2C - \frac{1}{k+1} + 2\alpha(d) - 2\beta(d) - 2\frac{\zeta'(2)}{\zeta(2)} \right) \right. \\ &\quad \left. + O\left(S(d)\left(\frac{x}{d}\right)^{k+\frac{1}{2}} \log \frac{x}{d}\right) \right) \\ &= \frac{x^{k+1}}{(k+1)\zeta(2)} \left(\left(\sum_{d \leq x} \frac{\mu^*(d)\phi(d)}{d^{k+1}\psi(d)} \right) \left(\log x + 2C - \frac{1}{k+1} - 2\frac{\zeta'(2)}{\zeta(2)} \right) - \sum_{d \leq x} \frac{\mu^*(d)\phi(d) \log d}{d^{k+1}\psi(d)} \right) \end{aligned}$$

$$\begin{aligned}
& +2 \sum_{d \leq x} \frac{\mu^*(d)\phi(d)\alpha(d)}{d^{k+1}\psi(d)} - 2 \sum_{d \leq x} \frac{\mu^*(d)\phi(d)\beta(d)}{d^{k+1}\psi(d)} \Big) + O \left(x^k \sqrt{x} \log x \sum_{d \leq x} \frac{S(d)}{d^{k+\frac{1}{2}}} \right) \\
& = \frac{x^{k+1}}{(k+1)\zeta(2)} \left(\alpha_k \left(\log x + 2C - \frac{1}{k+1} - 2 \frac{\zeta'(2)}{\zeta(2)} \right) + O \left(\log x \sum_{d > x} \frac{1}{d^{k+1}} \right) \right) \\
& - A_k + O \left(\sum_{d > x} \frac{\log d}{d^{k+1}} \right) + 2B_k + O \left(\sum_{d > x} \frac{\log d}{d^{k+1}} \right) - 2C_k + O \left(\sum_{d > x} \frac{1}{d^{k+1}} \right) \\
& + O \left(x^k \sqrt{x} \log x \sum_{d \leq x} \frac{S(d)}{d^{k+\frac{1}{2}}} \right).
\end{aligned}$$

Now using (3.6) and the well known estimates

$$\sum_{d > x} \frac{1}{d^{k+1}} = O \left(\frac{1}{x^k} \right), \quad \sum_{d > x} \frac{\log d}{d^{k+1}} = O \left(\frac{\log x}{x^k} \right)$$

the proof is complete.

Corollary 2. ($k = 1$) *We have*

$$\begin{aligned}
(3.8) \quad \sum_{n \leq x} P^*(n) & = \frac{3}{\pi^2} x^2 \left(\alpha \left(\log x + 2C - \frac{1}{2} - 2 \frac{\zeta'(2)}{\zeta(2)} \right) - A_1 + 2B_1 - 2C_1 \right) \\
& + O(x\sqrt{x} \log x),
\end{aligned}$$

where α is given by (1.5).

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