A GENERALIZATION OF PILLAI'S ARITHMETICAL FUNCTION INVOLVING REGULAR CONVOLUTIONS

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Abstract. We define a generalization of Pillai's arithmetical function $P(n) = \sum_{i=1}^{n}(i,n)$, in terms of Narkiewicz's regular convolutions. We give arithmetic evaluations for our new generalization of Pillai's function and we establish asymptotic formulae for it in case of cross-convolutions, investigated in our previous papers.

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1. Introduction

Pillai's ([8]) arithmetical function is defined by $P(n) = \sum_{i=1}^{n}(i,n)$, where (i,n) denotes the greatest common divisor (gcd) of i and n. In this paper we consider the following generalization of this function. Let A be a regular convolution of Narkiewicz-type ([7]) given by

$$(f *_A g)(n) = \sum_{d\in A(n)} f(d)g(n/d).$$

see also [6], [9], [16]. This is a common generalization of the Dirichlet convolution D and of the unitary convolution U.

We recall that if A is a regular convolution, then the elements of the set $A(n)$ are called the A-divisors of n and

(i) for every prime power p^a there exists a divisor $t = t_A(p^a)$ of a, called the type of p^a with respect to A, such that $A(p^{it}) = \{1, p^t, p^{2t}, ..., p^{it}\}$ for every $i \in \{0, 1, ..., a/t\}$,

(ii) the function I, defined by $I(n) = 1$ for all $n \in \mathbb{N}$, $\mathbb{N}$ denoting the set of positive integers, has an inverse μ_A with respect to the A-convolution, μ_A is multiplicative and for all prime powers p^a one has

$$\mu_A(p^a) = \begin{cases} -1, & \text{if } t_A(p^a) = a, \\ 0, & \text{otherwise.} \end{cases}$$

For $k \in \mathbb{N}$, let $A_k(n) = \{d \in \mathbb{N} : d^k \in A(n^k)\}$. The A_k-convolution is regular whenever the A-convolution is regular, see [9], Theorem 3.1. Let $(a,b)_{A,k}$ denote the largest k-th power divisor of a which belongs to $A(b)$. Note that $(a,b)_{D,k} \equiv (a,b)_k$ is the greatest common k-th power divisor of a and b.

Furthermore, let $u \in \mathbb{N}$, let $F = \{f_1, f_2, ..., f_u\}$ be a set of polynomials with integral coefficients and let g be an arbitrary arithmetical function. We define the generalized Pillai function $P^{(u)}_{F,A,k,g}$ by

$$(1) \qquad P^{(u)}_{F,A,k,g}(n) = \sum_{\substack{x_j \,(\mathrm{mod}\ n^k) \\ 1 \le j \le u}} g(((f_j(x_j)), n^k)_{A,k}),$$

where $(f_j(x_j))$ stands for the gcd of $f_1(x_1), ..., f_u(x_u)$.

We use the notations E_s, E and I for the functions $E_s(n) = n^s, E(n) \equiv E_1(n) = n$ and $I(n) \equiv E_0(n) = 1, n \in \mathbb{N}$, respectively.

For $A = D$, the function $P^{(u)}_{F,D,k,g} \equiv P^{(u)}_{F,k,g}$ was investigated by J. Chidambaraswamy and R. Sitaramachandrarao [2] and for $A = D, k = u = 1, f_1(x) = x$ and $g = E_r$ we get the function P_r defined by K. Alladi [1]. If $A = D, u = 1, f_1(x) = x$ and $g = E$ we obtain the function P_k introduced by H.G. Kopetzky [5], which reduces to the function P of S. S. Pillai [8] in case $k = 1$. The unitary analogues P^*_r (case $A = U, u = k = 1, f_1(x) = x, g = E_r$) and P^*_k (case $A = U, u = 1, f_1(x) = x, g = E$) were introduced and investigated by us in [13], [15].

For $A = D, k = 1, g = E$ and for polynomials of first degree $f_j(x) = s_j + (x - 1)d_j, (s_j, d_j) = 1, 1 \le j \le u$ the corresponding function was studied by us in [14].

We give arithmetical evaluations for our generalized Pillai function and we establish asymptotic formulae for it in the following three cases:

Case 1: $g = E_r$, F a set of nonconstant polynomials with an additional condition (including the case when all the polynomials are irreducible),

Case 2: $g = E_r$ with $r > u$ and $f_j(x) = s_j + (x-1)d_j^k, (s_j, d_j^k)_k = 1, 1 \le j \le u$,

Case 3: $g = E_u$ and $f_j(x) = s_j + (x-1)d_j^k, (s_j, d_j^k)_k = 1, 1 \le j \le u$,

assuming that A is a cross-convolution and using elementary arguments.

The notion of cross-convolution, as a special regular convolution was introduced in our previous papers [20], [16], [17], [18] as follows. We say that A is a *cross-convolution* if for every prime p we have either $t_A(p^a) = 1$, i.e. $A(p^a) = \{1, p, p^2, ..., p^a\} \equiv D(p^a)$ for every $a \in \mathbb{N}$ or $t_A(p^a) = a$, i.e. $A(p^a) = \{1, p^a\} \equiv U(p^a)$ for every $a \in \mathbb{N}$. Let P and Q be the sets of the primes of the first and second kind of above, respectively, where $P \cup Q = \mathbb{P}$ is the set of all primes. For $P = \mathbb{P}$ and $Q = \emptyset$ we have the Dirichlet convolution D and for $P = \emptyset$ and $Q = \mathbb{P}$ we obtain the unitary convolution U.

For $z > 1$ let

$$\zeta_P(z) = \prod_{p \in P} \left(1 - \frac{1}{p^z}\right)^{-1}, \qquad \zeta_Q(z) = \prod_{p \in Q} \left(1 - \frac{1}{p^z}\right)^{-1},$$

where $\zeta_P(z)\zeta_Q(z) = \zeta(z)$ is the Riemann zeta function.

Furthermore, let (P) and (Q) denote the multiplicative semigroups generated by $\{1\} \cup P$ and $\{1\} \cup Q$, respectively. Every $n \in \mathbb{N}$ can be written uniquely in the form $n = n_P n_Q$, where $n_P \in (P), n_Q \in (Q)$.

The results of this paper generalize and unify many known results concerning the special cases mentioned above.

2. Arithmetical evaluations

For a polynomial f with integral coefficients let $N_f(n)$ denote the number of incongruent solutions (mod n) of the congruence $f(x)\equiv 0 \pmod{n}$. It is well-known that the function N_f is multiplicative. Define the function N_F by $N_F(n)=N_{f_1}(n)N_{f_2}(n)...N_{f_u}(n)$ for each $n\in\mathbb{N}$. It follows that the function N_F is multiplicative.

The arithmetical evaluation of the function $P^{(u)}_{F,A,k,g}$ is given by

Theorem 1. *If A is a regular convolution, $F=\{f_1,f_2,...,f_u\}$ is an abitrary set of polynomials with integral coefficients, $k\in\mathbb{N}$ and g is an arithmetical function, then*

$$P^{(u)}_{F,A,k,g}=((g\circ E_k)*_{A_k}\mu_{A_k})(N_F\circ E_k)*_{A_k}E_{ku}, \tag{2}$$

where $\circ$ denotes the ordinary composition of functions.

If in addition g is multiplicative, then $P^{(u)}_{F,A,k,g}$ is multiplicative.

Proof. Grouping the terms of (1) according to the values $((f_j(x_j)),n^k)_{A,k}=d^k$ and using that $d^k\in A((a,b)_{A,k})$ if and only if $d^k|a$ and $d^k\in A(b)$, see [9], Theorems 4.2 and 4.3, we get

$$P^{(u)}_{F,A,k,g}(n)=\sum_{d\in A_k(n)}\sum_{\substack{x_j \,(\mathrm{mod}\ n^k)\\ 1\le j\le u\\ ((f_j(x_j)),n^k)_{A,k}=d^k}} g(d^k)=\sum_{d\in A_k(n)} g(d^k)T_d, \qquad \text{where}$$

$$T_d=\sum_{\substack{x_j \,(\mathrm{mod}\ n^k)\\ 1\le j\le u\\ ((f_j(x_j)),n^k)_{A,k}=d^k}} 1=\sum_{\substack{x_j \,(\mathrm{mod}\ n^k)\\ 1\le j\le u\\ ((f_j(x_j)/d^k),(n/d)^k)_{A,k}=1}} 1$$

$$=\sum_{\substack{x_j \,(\mathrm{mod}\ n^k)\\ 1\le j\le u}}\sum_{e^k\in A(((f_j(x_j)/d^k),(n/d)^k)_{A,k})}\mu_{A_k}(e)=\sum_{\substack{x_j \,(\mathrm{mod}\ n^k)\\ 1\le j\le u}} 1 \sum_{\substack{e^k|f_j(x_j)/d^k\\ e\in A_k(n/d)}}\mu_{A_k}(e)$$

$$=\sum_{e\in A_k(n/d)}\mu_{A_k}(e)\sum_{\substack{x_j \,(\mathrm{mod}\ n^k)\\ 1\le j\le u\\ f_j(x_j)\equiv 0 (\mathrm{mod}\ (de)^k)}} 1=\sum_{e\in A_k(n/d)}\mu_{A_k}(e)(\frac{n}{de})^{ku}N_F((de)^k).$$

Hence

$$P^{(u)}_{F,A,k,g}(n)=n^{ku}\sum_{d\in A_k(n)} g(d^k)\sum_{e\in A_k(n/d)}\mu_{A_k}(e)\frac{N_F((de)^k)}{(de)^{ku}}.$$

Denoting $de=\delta\in A_k(n)$, where $d\in A_k(n), e\in A_k(n/d)$ if and only if $\delta\in A_k(n), d\in A_k(\delta)$, cf. [9], Theorem 2.1, we have

$$P^{(u)}_{F,A,k,g}(n)=n^{ku}\sum_{\delta\in A_k(n)}\frac{N_F(\delta^k)}{\delta^{ku}}\sum_{d\in A_k(\delta)} g(d^k)\mu_{A_k}(\delta/d),$$

which finishes the proof of (2). It has been already noted that N_F is multiplicative. If g is multiplicative, then using that regular convolutions preserve the multiplicativity, we get that $P^{(u)}_{F,A,k,g}$ is multiplicative.

Let $\phi_{A,s} = \mu_A *_A E_s$. For $s = ku$ and for A_k instead of A,

$$\phi_{A_k,ku}(n) \equiv \phi^{(u)}_{A,k}(n) = (\mu_{A_k} *_{A_k} E_{ku})(n) \tag{3}$$

represents the number of ordered u-tuples $\langle x_1, x_2, ..., x_u\rangle$ (mod n^k) such that $((x_j), n^k)_{A,k}) = 1$. This generalized Euler function was introduced by P. Haukkanen and P. J. McCarthy [4], see also [3]. Observe that $\phi_{D,1} \equiv \phi$ is the Euler function.

Corollary 1. ($g = E_r$ and $g = E_u$)

$$P^{(u)}_{F,A,k,E_r} \equiv P^{(u)}_{F,A,k,r} = \phi_{A_k,rk}(N_F \circ E_k) *_{A_k} E_{ku},$$

$$P^{(u)}_{F,A,k,u} = \phi^{(u)}_{A,k}(N_F \circ E_k) *_{A_k} E_{ku}.$$

If $f_j(x) = s_j + (x-1)d_j^k$, $j = 1, 2, ..., u$, then let $P^{(u)}_{F,A,k,g} \equiv P^{(u)}_{A,k,g}(\mathbf{s}, \mathbf{d}, .)$, where $\mathbf{s} = \langle s_1, s_2, ..., s_u\rangle$ and $\mathbf{d} = \langle d_1, d_2, ..., d_u\rangle$. Taking into account that in this case $N_{f_j}(n) = (d_j^k, n)$ if $(d_j^k, n)|s_j$ and $N_{f_i}(n) = 0$ otherwise, from Theorem 1 we get the following result.

Corollary 2. *For every $A, g, k, u, \mathbf{s}, \mathbf{d}$ and $n \in \mathbb{N}$ we have*

$$P^{(u)}_{A,k,g}(\mathbf{s}, \mathbf{d}, n) = n^{ku} \sum_{\substack{e \in A_k(n) \\ (e,d_j)^k | s_j \\ 1 \le j \le u}} ((g \circ E_k) *_{A_k} \mu_{A_k})(e) e^{-ku} (e, d_1)^k (e, d_2)^k ... (e, d_u)^k.$$

Let $\delta = d_1 d_2 ... d_u$. We have

Corollary 3. *If $(s_j, d_j^k)_k = 1$, $j = 1, 2, ..., u$, then for every $n \in \mathbb{N}$,*

$$P^{(u)}_{A,k,g}(\mathbf{s}, \mathbf{d}, n) = n^{ku} \sum_{\substack{e \in A_k(n) \\ (e,\delta)=1}} ((g \circ E_k) *_{A_k} \mu_{A_k})(e) e^{-ku}$$

and if in addition g is a multiplicative arithmetical function, then

$$P^{(u)}_{A,k,g}(\mathbf{s}, \mathbf{d}, n) = n^{ku} \prod_{\substack{p^a || n \\ (p,\delta)=1}} \left(\frac{g(p^{ak})}{p^{aku}} + \left(1 - \frac{1}{p^{kut}}\right) \sum_{i=0}^{a/t-1} \frac{g(p^{kit})}{p^{kuit}} \right).$$

for every $n \in \mathbb{N}, n > 1$, where $p^a || n$ means $p^a | n, p^{a+1} \nmid n$ and $t = t_{A_k}(p^a)$.

Proof. Since $(s_j, d_j^k)_k = 1$, we have $(e, d_j)^k | s_j$ if and only if $(e, d_j) = 1$. Furthermore for $n = p^a$, with $p \nmid d_j, 1 \le j \le u$ and $A_k(p^a) = \{1, p^t, p^{2t}, ..., p^a\}, t = t_{A_k}(p^a)$ we have

$$P^{(u)}_{A,k,g}(\mathbf{s}, \mathbf{d}, p^a) = \sum_{i=0}^{a/t} g(p^{ikt}) \phi^{(u)}_{A,k}(p^{a-it}).$$

Using now that $\phi^{(u)}_{A,k}(1)=1$ and

$$\phi^{(u)}_{A,k}(p^{a-it})=(p^{ku})^{a-it}(1-\frac{1}{p^{kut}}),$$

for every $i\in\{0,1,...,a/t-1\}$, we get

$$P^{(u)}_{A,k,g}(\mathbf{s},\mathbf{d},p^a)=g(p^{ak})+\sum_{i=0}^{a/t-1}g(p^{kit})(p^{ku})^{a-it}(1-\frac{1}{p^{kut}}).$$

If $n=p^a$ and $p|d_j$ for some $j, 1\le j\le u$, then $P^{(u)}_{A,k,g}(\mathbf{s},\mathbf{d},p^a)=p^{aku}$ and the proof is complete.

Corollary 4. $(g=E_r)$ *If* $r\ne u$, *then*

$$P^{(u)}_{A,k,r}(\mathbf{s},\mathbf{d},n)=n^{ku}\prod_{\substack{p^a||n\\(p,\delta)=1}}\left(p^{ak(r-u)}+\left(1-\frac{1}{p^{kut}}\right)\frac{p^{ak(r-u)}-1}{p^{tk(r-u)}-1}\right),$$

and if $r=u$, *then*

$$P^{(u)}_{A,k,u}(\mathbf{s},\mathbf{d},n)=n^{ku}\prod_{\substack{p^a||n\\(p,\delta)=1}}\left(1+\left(1-\frac{1}{p^{kut}}\right)\frac{a}{t}\right),$$

for every $n\in\mathbb{N}, n>1$, *where* $t=t_{A_k}(p^a)$.

Corollary 5. *If A is a cross-convolution, then for every* $n\in\mathbb{N}, n>1$ *we have*

$$P^{(u)}_{A,k,u}(\mathbf{s},\mathbf{d},n)=n^{ku}\prod_{\substack{p^a||n,p\in P\\(p,\delta)=1}}\left(1+a\left(1-\frac{1}{p^{ku}}\right)\right)\prod_{\substack{p^a||n,p\in Q\\(p,\delta)=1}}\left(2-\frac{1}{p^{aku}}\right).$$

For $s_j=d_j=1$, i.e. for $f_j(x)=x, 1\le j\le u$, let $P^{(u)}_{F,A,k,g}\equiv P^{(u)}_{A,k,g}$ and we get from Theorem 1

Corollary 6. *For every regular convolution A and for every g,k,u we have*

$$P^{(u)}_{A,k,g}=(g\circ E_k)*_{A_k}\phi^{(u)}_{A,k},$$

$$P^{(u)}_{A,k,E_r}\equiv P^{(u)}_{A,k,r}=E_{kr}*_{A_k}\phi^{(u)}_{A,k}.$$

Remark 1. If A is a cross-convolution, then $A_k=A$ for every $k\in\mathbb{N}$, see [9], Theorem 3.3, [16], Remark 2 and from (3) we have

$$\phi^{(u)}_{A,k}=\mu_A*_A E_{ku}=\phi^{(k)}_{A,u}=\phi^{(1)}_{A,ku}=\phi^{(ku)}_{A,1},$$

$$P^{(u)}_{A,k,g}=(g\circ E_k)*_A\phi^{(u)}_{A,k},\quad P^{(u)}_{A,k,r}=E_{rk}*_A\phi^{(u)}_{A,k}$$

and it follows that

$$P^{(u)}_{A,k,u}=E_{ku}*_A\phi^{(u)}_{A,k}=P^{(k)}_{A,u,k}=P^{(1)}_{A,ku,1}. \tag{4}$$

Another representation of the function $P^{(u)}_{A,k,u}(\mathbf{s},\mathbf{d},.)$ is given by

Theorem 2. *If* $(s_j, d_j^k)_k = 1$, $j = 1, 2, ..., u$, *then*

$$P^{(u)}_{A,k,u}(\mathbf{s}, \mathbf{d}, .) = \mu_{A_k} I_\delta *_{A_k} E_{ku} \tau_{A_k}(., \delta),$$

where $I_\delta(n) = 1$ *or* 0, *according as* n *and* δ *are coprime or not, and* $\tau_A(n, \delta)$ *denotes the number of* A*-divisors of* n *which are prime to* δ.

Proof. We deduce from Corollary 3

$$P^{(u)}_{A,k,u}(\mathbf{s}, \mathbf{d}, .) = (E_{ku} *_{A_k} \mu_{A,k}) I_\delta *_{A_k} E_{ku} = \mu_{A_k} I_\delta *_{A_k} E_{ku} I_\delta *_{A_k} E_{ku}$$
$$= \mu_{A_k} I_\delta *_{A_k} E_{ku}(I_\delta *_{A_k} I) = \mu_{A_k} I_\delta *_{A_k} E_{ku} \tau_{A_k}(., \delta).$$

For other special choices of g we have for example the following results.

Theorem 3. *For every regular convolution* A *and for every* $n \in \mathbb{N}$,

$$\sum_{\substack{x_j \pmod{n} \\ 1 \le j \le u}} \sigma_{A,u}(((x_j), n)_A) = n^u \tau_A(n), \tag{5}$$

$$\sum_{\substack{x_j \pmod{n} \\ 1 \le j \le u}} \tau_A(((x_j), n)_A) = \sigma_{A,u}(n), \tag{6}$$

$$\sum_{\substack{x_j \pmod{n} \\ 1 \le j \le u}} z^{\omega((x_j), n)} = n^u \prod_{p|n} (1 + \frac{z-1}{p^u}), \tag{7}$$

where $\tau_A(n)$ *and* $\sigma_{A,u}(n)$ *denote the number of* A*-divisors of* n *and the sum of* u*-th powers of* A*-divisors of* n, *respectively,* $\omega(n)$ *is the number of distinct prime factors of* n *and* z *is a complex number.*

Proof. In case $k = 1$ using Corollary 6 we deduce

$$P^{(u)}_{A,1,g} = g *_A \phi^{(u)}_{A,1} = g *_A \mu_A *_A E_u.$$

Now for $g = \sigma_{A,u} = I *_A E_u$ we get

$$P^{(u)}_{A,1,\sigma_{A,u}} = I *_A E_u *_A \mu_A *_A E_u = E_u *_A E_u = E_u \tau_A,$$

which is relation (5), and for $g = \tau_A = I *_A I$ we conclude

$$P^{(u)}_{A,1,\tau_A} = I *_A I *_A \mu_A *_A E_u = I *_A E_u = \sigma_{A,u},$$

giving (6). Finally, for $k = 1, A = D$ and $g(n) = z^{\omega(n)}$ we have

$$P^{(u)}_{D,1,g}(n) = (g * \phi^{(u)}_{D,1})(n) = n^u \prod_{p|n} (1 + \frac{z-1}{p^u}),$$

where the last equality can be easily obtained using the multiplicativity of the involved functions.

For $u = 1$ and $z = 2$ relation (7) is due to us, see [11]. The function $\psi_u(n) = n^u \prod_{p|n}(1 + \frac{1}{p^u})$ is the generalized Dedekind function defined by D. Suryanarayana [10].

Remark 2. If g is a real valued increasing function and A and B are two regular convolutions such that $A(n) \subseteq B(n)$ for every $n \in \mathbb{N}$, then $P^{(u)}_{F,A,k,g}(n) \le P^{(u)}_{F,B,k,g}(n)$, for every $n \in \mathbb{N}$. In particular, $P^*_k(n) \le P^{(1)}_{A,k,E}(n) \le P_k(n)$ for every regular convolution A and for every $n \in \mathbb{N}$.

3. Asymptotic formulae

We need the following lemmas.

Lemma 1.

(8)
$$\sum_{n\le x} n^{-s} = \begin{cases} O(x^{1-s}), & 0<s<1,\\ O(\log x), & s=1,\\ O(1), & s>1,\end{cases}$$

(9)
$$\sum_{n>x} n^{-s} = O(x^{1-s}), \quad s>1.$$

Lemma 2. (cf. [20], Lemma 8) *If A is a cross-convolution, $s\ge 0$ and $a\in\mathbb{N}$, then*

$$\sum_{\substack{n\le x\\ (n,a)\in(P)}} n^s = \frac{\phi(a_Q)x^{s+1}}{a_Q(s+1)} + O(x^{s+\varepsilon} f_A(a)),$$

where $f_A(a)=1$ or $f_A(a)=\sigma^*_{-\varepsilon}(a)$ the sum of $(-\varepsilon)$-th powers of the unitary divisors of a, according as the set Q is finite or Q is infinite, respectively for every $0\le\varepsilon<1$.

Case 1: We consider first the function $P^{(u)}_{F,A,k,r}$ obtained for $g=E_r$.

Let f be a nonconstant polynomial with integral coefficients and let its decomposition into irreducible factors be $f=cg_1^{r_1}g_2^{r_2}...g_m^{r_m}$. Define $h(f)=\max_{1\le j\le m} r_j$.

Lemma 3. ([20], Lemma 6) *For every set F of nonconstant polynomials and for every $\varepsilon>0$ we have*

$$N_F(n)=O(n^{u-h+\varepsilon}),$$

where $h=1/h(f_1)+1/h(f_2)+...+1/h(f_u)$.

Theorem 4. *If A is a cross-convolution, F is an arbitrary set of nonconstant polynomials, $k,u\in\mathbb{N}$ and $0<r<h$, then*

(10)
$$\sum_{n\le x} P^{(u)}_{F,A,k,r}(n) = \frac{x^{ku+1}}{ku+1}\sum_{n=1}^{\infty}\frac{\phi_{A,kr}(n)N_F(n^k)\phi(n_Q)}{n^{ku+1}n_Q} + O(R(x)),$$

where $R(x)=x^{ku}$ if $h>r+\frac{1}{k}$ and $R(x)=x^{ku+1-k(h-r)+\varepsilon}$ if $h\le r+\frac{1}{k}$ for every $0<\varepsilon<k(h-r)$.

Proof. Using Corollary 1 and Lemma 2 with $\varepsilon=0$ and using that $\tau(n)=O(n^\varepsilon)$ for every $\varepsilon>0$, where $\tau(n)$ is the divisor function,

$$\sum_{n\le x} P^{(u)}_{F,A,k,r}(n) = \sum_{\substack{de=n\le x\\ (d,e)\in(P)}} \phi_{A,kr}(d)N_F(d^k)e^{ku} = \sum_{d\le x}\phi_{A,kr}(d)N_F(d^k)\sum_{\substack{e\le x/d\\ (e,d)\in(P)}} e^{ku}$$

$$=\sum_{d\le x}\phi_{A,kr}(d)N_F(d^k)\left(\frac{\phi(d_Q)}{(ku+1)d_Q}\left(\frac{x}{d}\right)^{ku+1} + O\left(\left(\frac{x}{d}\right)^{ku} d^\varepsilon\right)\right)$$

$$= \frac{x^{ku+1}}{ku+1} \sum_{d\le x} \frac{\phi_{A,kr}(d) N_F(d^k)\phi(d_Q)}{d^{ku+1} d_Q} + O\left(x^{ku} \sum_{d\le x} \frac{\phi_{A,kr}(d) N_F(d^k)}{d^{ku-\varepsilon}} \right)$$

$$= \frac{x^{ku+1}}{ku+1} \sum_{d=1}^{\infty} \frac{\phi_{A,kr}(d) N_F(d^k)\phi(d_Q)}{d^{ku+1} d_Q} + O\left(x^{ku+1} \sum_{d> x} \frac{N_F(d^k)}{d^{ku+1-kr}} \right) + O\left(x^{ku} \sum_{d\le x} \frac{N_F(d^k)}{d^{ku-\varepsilon-kr}} \right),$$

using that $\phi_{A,kr}(d) \le d^{kr}$ for every $d \in \mathbb{N}$. Here the series of the main term is absolutely convergent, since its general term is

$$O\left(\frac{d^{kr} d^{k(u-h)+\varepsilon}}{d^{ku+1}} \right) = O\left(\frac{1}{d^{1+k(h-r)-\varepsilon}} \right),$$

applying Lemma 3 and choosing $\varepsilon < k(h-r)$. The first O-term is

$$O\left(x^{ku+1} \sum_{d>x} \frac{1}{d^{1+k(h-r)-\varepsilon}} \right) = O\left(x^{ku+1} \frac{1}{d^{k(h-r)-\varepsilon}} \right) = O(x^{ku+1-k(h-r)+\varepsilon})$$

using (9) with $\varepsilon < k(h-r)$.

The second O-term is

$$O\left(x^{ku} \sum_{d\le x} \frac{d^{k(u-h)+\varepsilon/2}}{d^{ku-\varepsilon/2-kr}} \right) = O\left(x^{ku} \sum_{d\le x} \frac{1}{d^{k(h-r)-\varepsilon}} \right),$$

by Lemma 3, which is, using (8): $O(x^{ku})$ for $k(h-r) > 1$, choosing $\varepsilon < k(h-r)-1$ and $O(x^{ku+1-k(h-r)+\varepsilon})$ for $0 < k(h-r) \le 1$ with $\varepsilon < k(h-r)$, and the proof is complete.

Corollary 7. *If A is a cross-convolution, F is an arbitrary set of nonconstant irreducible polynomials, $k, u \in \mathbb{N}$ and $0 < r < u$, then (10) holds with the error term $R(x) = x^{ku}$ if $u > r + \frac{1}{k}$ and $R(x) = x^{kr+1+\varepsilon}$ if $u \le r + \frac{1}{k}$ for every $0 < \varepsilon < k(u-r)$.*

Proof. In case of irreducible polynomials f_i we have $h(f_i) = 1$, thus $h = u$ and we apply Theorem 4.

For $A = D$ the result of Corollary 7 was proved in [2], Theorem 3.2.

Case 2: Next we consider the function $P^{(u)}_{A,k,r}(\mathbf{s}, \mathbf{d}, .)$ obtained for $g = E_r, r > u$ and $f_j(x) = s_j + (x-1)d_j^k$, where $(s_j, d_j^k)_k = 1, 1 \le j \le u$.

Lemma 4. (see [12], Lemma 5)

$$\sum_{n\le x} \frac{\tau(n)}{n^s} = \begin{cases} O(x^{1-s}\log x), & 0 < s < 1, \\ O(\log^2 x), & s = 1, \\ O(1), & s > 1. \end{cases} \tag{11}$$

Theorem 5. *If A is a cross-convolution, $k,u\in\mathbb{N}, r>u$ and $(s_j,d_j^k)_k=1, 1\le j\le u$, then*

$$\sum_{n\le x} P^{(u)}_{A,k,r}(\mathbf{s},\mathbf{d},n)=\frac{\Delta\phi(\delta)x^{kr+1}}{\delta(kr+1)}+O(S(x)),$$

where Δ is given by

$$\Delta=\frac{\zeta(k(r-u)+1)\zeta_Q(kr+1)}{\zeta_P(kr+1)}\prod_{p|\delta_P}\left(1-\frac{1}{p^{kr+1}}\right)^{-1}\prod_{p|\delta_Q}\left(1-\frac{1}{p^{kr+1}}\right)\times$$

$$\prod_{\substack{p\in Q\\(p,\delta)=1}}\left(1-\frac{2}{p^{kr+1}}+\frac{1}{p^{kr+2}}-\frac{1}{p^{k(r-u)+2}}+\frac{1}{p^{k(2r-u)+2}}\right), \tag{12}$$

and $S(x)=x^{kr}(r>u+\frac{1}{k}), x^{kr}\log^2 x(r=u+\frac{1}{k}, Q$ infinite $), x^{kr}\log x(r=u+\frac{1}{k}, Q$ finite $), x^{ku+1}(r<u+\frac{1}{k})$.

Proof. By Corollary 3 we have $P^{(u)}_{A,k,r}(\mathbf{s},\mathbf{d},.)=(E_{kr}*_A\mu_A)I_\delta*_A E_{ku}=E_{kr}I_\delta*_A\mu_A I_\delta*_A E_{ku}=h*_A E_{kr}I_\delta$, where $h=E_{ku}*_A\mu_A I_\delta$. Now from Lemma 2, for every $0\le\varepsilon<1$,

$$\sum_{n\le x} P^{(u)}_{A,k,r}(\mathbf{s},\mathbf{d},n)=\sum_{e\le x}h(e)\sum_{\substack{j\le x/e\\(j,\delta e_Q)=1}} j^{kr}$$

$$=\sum_{e\le x}h(e)\left(\frac{\phi(\delta e_Q)}{(kr+1)\delta e_Q}\left(\frac{x}{e}\right)^{kr+1}+O\left(\left(\frac{x}{e}\right)^{kr+\varepsilon}f_A(\delta e_Q)\right)\right)$$

$$=\frac{x^{kr+1}}{kr+1}\cdot\frac{\phi(\delta)}{\delta}\sum_{e\le x}\frac{h(e)f(e_Q,\delta)}{e^{kr+1}}+O\left(x^{kr+\varepsilon}\sum_{e\le x}\frac{e^{ku}f_A(\delta e_Q)}{e^{kr+\varepsilon}}\right),$$

where

$$f(n,\delta)=\prod_{\substack{p|n\\(p,\delta)=1}}\left(1-\frac{1}{p}\right)\quad\text{and}\quad h(n)\le n^{ku}.$$

Hence we obtain

$$\sum_{n\le x} P^{(u)}_{A,k,r}(\mathbf{s},\mathbf{d},n)=\frac{\phi(\delta)x^{kr+1}}{\delta(kr+1)}\sum_{e=1}^{\infty}\frac{h(e)f(e_Q,\delta)}{e^{kr+1}}+O\left(x^{kr+1}\sum_{e>x}\frac{e^{ku}}{e^{kr+1}}\right)$$

$$+O\left(x^{kr+\varepsilon}\sum_{e\le x}\frac{f_A(\delta e_Q)}{e^{k(r-u)+\varepsilon}}\right).$$

Here the series is absolutely convergent, since its general term is $O(1/n^{k(r-u)+1})$, where $r-u>0$. Let Δ be the sum of the series. The general term is multiplicative in e and using Euler's product formula we get (12) for Δ.

The first O-term is $O(x^{ku+1})$ by (9) and the second O-tem is for Q finite and choosing $\varepsilon = 0$: $O(x^{kr})$ for $k(r-u) > 1$; $O(x^{kr}\log x)$ for $k(r-u) = 1$; $O(x^{kr}\cdot x^{-k(r-u)+1}) = O(x^{ku+1})$ for $k(r-u) < 1$, applying (8).

Furthermore, for Q infinite the second O-term is using (11): $O(x^{kr})$ if $k(r-u) > 1$ with $\varepsilon = 0$; $O(x^{kr}\log^2 x)$ if $k(r-u) = 1$ with $\varepsilon = 0$; and if $k(r-u) < 1$ and selecting $0 < \varepsilon < 1 - k(r-u)$ it is

$$O\left(x^{kr+\varepsilon}\sum_{e\le x}\frac{\sigma^*_{-\varepsilon}(e)}{e^{k(r-u)+\varepsilon}}\right) = O(x^{ku+1}),$$

see [13], Lemma 2.2.

Corollary 8. ($f_j(x) = x, 1 \le j \le u, \delta = 1$) *If A is a cross-convolution and $k, u \in \mathbb{N}, r > u$ then*

$$\sum_{n\le x} P^{(u)}_{A,k,r}(n) = \frac{\Theta x^{kr+1}}{kr+1} + O(S(x)),$$

where

$$\Theta = \frac{\zeta(k(r-u)+1)\zeta_Q(kr+1)}{\zeta_P(kr+1)}\prod_{p\in Q}\left(1 - \frac{2}{p^{kr+1}} + \frac{1}{p^{kr+2}} - \frac{1}{p^{k(r-u)+2}} + \frac{1}{p^{k(2r-u)+2}}\right)$$

and $S(x)$ is defined in Theorem 5.

For $A = D$ this result is due in [2], Theorem 3.2.

In case $A = U, k = u = 1$ the result of Corollary 8 is proved in [13], Theorem 4.2.

Case 3: Now we deal with the function $P^{(u)}_{A,k,u}(\mathbf{s},\mathbf{d},.)$ obtained for $g = E_u$ and $f_j(x) = s_j + (x-1)d_j^k, (s_j, d_j^k)_k = 1, 1 \le j \le u$.

We also need the following lemmas.

Lemma 5. ([19]) *If A is a cross-convolution and $u, t \in \mathbb{N}$, then*

$$\sum_{\substack{n\le x\\(n,u)=1}} \tau_A(n,t) = \left(\frac{\phi(u)}{u}\right)^2 f(t,u)\frac{(tu)_Q^2}{\zeta_Q(2)\phi_2((tu)_Q)}x\left(\log x + 2C - 1 + 2\alpha(u) + \alpha(t,u)\right.$$

$$\left.-2\beta((tu)_Q) - 2\frac{\zeta'_Q(2)}{\zeta_Q(2)}\right) + O(\sigma^*_{-1/2}(t,u)S(u)H(x,Q)), \tag{13}$$

where C is Euler's constant,

$$f(t,u) = \prod_{\substack{p|t\\(p,u)=1}}\left(1-\frac{1}{p}\right), \qquad \phi_2(n) = n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right), \qquad \alpha(t,u) = \sum_{\substack{p|t\\(p,u)=1}}\frac{\log p}{p-1},$$

$$\alpha(u) \equiv \alpha(u,1) = \sum_{p|u}\frac{\log p}{p-1}, \qquad \beta(u) = \sum_{p|u}\frac{\log p}{p^2-1}, \qquad S(u) = \sum_{d|u}\frac{3^{\omega(d)}}{\sqrt{d}}$$

$\zeta'_Q(s)$ is the derivative of $\zeta_Q(s)$, $\sigma^*_s(t,u)$ is the sum of s-th powers of the unitary divisors of t which are prime to u and $H(x,Q) = \sqrt{x}$ (Q finite), $\sqrt{x}\log x$ (Q infinite).

Lemma 6. *If A is a cross-convolution and $u,t\in\mathbb{N}$, then*

$$\sum_{\substack{n\le x\\(n,u)\in(P)}}\tau_A(n,t)=\frac{F_A(t,u)}{\zeta_Q(2)}x\left(\log x+2C-1+2\alpha(u_Q)+\alpha(t,u_Q)-2\beta(t_Qu_Q)\right.$$

$$\left.-2\frac{\zeta_Q'(2)}{\zeta_Q(2)}\right)+O(\sigma^*_{-1/2}(t,u_Q)S(u_Q)H(x,Q)),$$

where

$$F_A(t,u)=\frac{(\phi(u_Q))^2f(t,u_Q)t_Q^2}{\phi_2(t_Qu_Q)}.$$

Proof. Apply (13) for u_Q instead of u.

Remark 3. For every cross-convolution A and every $u,t\in\mathbb{N}$ we have $0<F_A(t,u)\le 1$.

Lemma 7. *If A is a cross-convolution and $u,t,b\in\mathbb{N}$, then*

$$\sum_{\substack{n\le x\\(n,u)\in(P)}}n^b\tau_A(n,t)=\frac{F_A(t,u)x^{b+1}}{(b+1)\zeta_Q(2)}\left(\log x+2C-\frac{1}{b+1}+2\alpha(u_Q)+\alpha(t,u_Q)-2\beta(t_Qu_Q)\right.$$

$$\left.-2\frac{\zeta_Q'(2)}{\zeta_Q(2)}\right)+O(\sigma^*_{-1/2}(t,u_Q)S(u_Q)J_b(x,Q)),$$

where $J_b(x,Q)=x^b\sqrt{x}$ (Q finite), $x^b\sqrt{x}\log x$ (Q infinite).

Proof. By partial summation from Lemma 6.

Lemma 8. *If A is a cross-convolution, $t\in\mathbb{N}$ and $s>0$, then the series*

$$\sum_{\substack{n=1\\(n,t)=1}}^{\infty}\frac{\mu_A(n)F_A(t,n)}{n^{s+1}},\qquad\sum_{\substack{n=1\\(n,t)=1}}^{\infty}\frac{\mu_A(n)F_A(t,n)\log n}{n^{s+1}},$$

$$\sum_{\substack{n=1\\(n,t)=1}}^{\infty}\frac{\mu_A(n)F_A(t,n)(2\alpha(n_Q)+\alpha(t,n_Q)-2\beta(n_Qt_Q))}{n^{s+1}}$$

are absolutely convergent. Let $A_t(s),B_t(s)=-A_t'(s)$ (derivative with respect to s) and $C_t(s)$ denote their sums.

Proof. The absolute convergence follows at once by Remark 3 and by $\alpha(n)=O(\log n)$, $\beta(n)=O(1)$.

Theorem 6. *If A is a cross-convolution, $k, u \in \mathbb{N}$ and $(s_j, d_j^k)_k = 1, 1 \leq j \leq u$, then*

$$\sum_{n \leq x} P_{A,k,u}^{(u)}(\mathbf{s}, \mathbf{d}, n) = \frac{x^{ku+1}}{(ku+1)\zeta_Q(2)} \left(A_\delta(ku) \left(\log x + 2C - \frac{1}{ku+1} - 2\frac{\zeta_Q'(2)}{\zeta_Q(2)} \right) - B_\delta(ku) \right.$$

$$\left. + C_\delta(ku) \right) + O(J_{ku}(x, Q)),$$

where $A_\delta(ku), B_\delta(ku), C_\delta(ku)$ and $J_{ku}(x, Q)$ are defined in Lemma 8 and Lemma 7, repectively.

Proof. Using Theorem 2 and lemma 7 with $b = ku, u = d, t = \delta$ we get

$$\sum_{n \leq x} P_{A,k,u}^{(u)}(\mathbf{s}, \mathbf{d}, n) = \sum_{\substack{d \leq x \\ (d,\delta)=1}} \mu_A(d) \sum_{\substack{e \leq x/d \\ (e,d) \in (P)}} e^{ku} \tau_A(e, \delta) =$$

$$= \sum_{\substack{d \leq x \\ (d,\delta)=1}} \mu_A(d) \left(\frac{F_A(\delta, d) x^{ku+1}}{(ku+1)\zeta_Q(2) d^{ku+1}} \left(\log \frac{x}{d} + 2C - \frac{1}{ku+1} + 2\alpha(d_Q) + \alpha(\delta, d_Q) \right. \right.$$

$$\left. \left. -2\beta(d_Q \delta_Q) \; -2\frac{\zeta_Q'(2)}{\zeta_Q(2)} \right) + O\left(\sigma_{-1/2}^*(\delta, d_Q) S(d_Q) J_{ku}(\frac{x}{d}, Q) \right) \right)$$

$$= \frac{x^{ku+1}}{(ku+1)\zeta_Q(2)} \left(\left(\left(\sum_{\substack{d \leq x \\ (d,\delta)=1}} \frac{\mu_A(d) F_A(\delta, d)}{d^{ku+1}} \right) \left(\log x + 2C - \frac{1}{ku+1} - 2\frac{\zeta_Q'(2)}{\zeta_Q(2)} \right) \right. \right.$$

$$\left. - \sum_{\substack{d \leq x \\ (d,\delta)}} \frac{\mu_A(d) F_A(\delta, d) \log d}{d^{ku+1}} + \sum_{\substack{d \leq x \\ (d,\delta)=1}} \frac{\mu_A(d) F_A(\delta, d)(2\alpha(d_Q) + \alpha(\delta, d_Q) - 2\beta(d_Q \delta_Q))}{d^{ku+1}} \right)$$

$$+ O\left(\sum_{\substack{d \leq x \\ (d,\delta)=1}} \sigma_{-1/2}^*(\delta, d_Q) S(d_Q) J_{ku}(\frac{x}{d}, Q) \right)$$

$$= \frac{x^{ku+1}}{(ku+1)\zeta_Q(2)} \left(A_\delta(ku) \left(\log x + 2C - \frac{1}{ku+1} - 2\frac{\zeta_Q'(2)}{\zeta_Q(2)} \right) + O\left(\log x \sum_{d > x} \frac{1}{d^{ku+1}} \right) \right.$$

$$\left. -B_\delta(ku) + C_\delta(ku) + O\left(\sum_{d > x} \frac{\log d}{d^{ku+1}} \right) + O\left(x^{ku+1/2} (\log x)^\gamma \sum_{d \leq x} \frac{S(d)}{d^{ku+1/2}} \right) \right),$$

where $\gamma = 0$ if Q is finite and $\gamma = 1$ if Q is infinite. Now using (8), the well-known estimate

$$\sum_{d > x} \frac{\log d}{d^s} = O\left(\frac{\log x}{x^{s-1}} \right), \quad s > 1, \qquad \text{and} \qquad \sum_{n \leq x} \frac{S(n)}{n^{s+1/2}} = O(1), \quad s \geq 1,$$

see [15], Proposition 7, the proof is complete.

Remark 4. For $f_j(x) = x, 1 \le j \le u$ we have $\delta = 1$,

$$A_1(ku) = \frac{1}{\zeta_P(ku+1)} \prod_{p \in Q} \left(1 - \frac{p-1}{(p+1)(p^{ku+1}-1)}\right),$$

$$B_1(ku) = A_1(ku) \left(\frac{\zeta_P'(ku+1)}{\zeta_P(ku+1)} - \sum_{p \in Q} \frac{(p-1)p^{ku+1}\log p}{(p+1)(p^{ku+1}-1)^2} \left(1 - \frac{p-1}{(p+1)(p^{ku+1}-1)}\right)^{-1} \right).$$

For $A = D, u = 1, \delta = 1$ this result is due in [2], Theorem 3.1. In case $A = U, u = 1, \delta = 1$ the result of Theorem 6 is proved in [15].

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