On an operation involving regular convolutions

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Abstract

We introduce operation (1.3) which is a generalization of the regular A-convolutions and has properties analogous to the operation (1.4). As an application we investigate certain Euler-type functions and Pillai-type functions.

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1 Introduction

Let $\mathbf{N}, \mathbf{R}$ and $\mathbf{C}$ denote the sets of positive integers, real numbers and complex numbers, respectively, and denote by $\mathcal{A}$ the set of arithmetical functions $f : \mathbf{N} \to \mathbf{C}$. For $u \in \mathbf{N} \cup \{0\}$ let $\mathcal{F}_u$ be the set of functions of $u+1$ variables $F : \mathbf{N} \times \mathbf{C}^u \to \mathbf{C}$. The values of F will be denoted by $F(n, \mathbf{x})$, where $\mathbf{x} = \langle x_1, ..., x_u \rangle \in \mathbf{C}^u$.

For $f \in \mathcal{A}$ and $F \in \mathcal{F}_u$ we introduce the operation $\circ$ by

$$(1.1) \qquad (f \circ F)(n, \mathbf{x}) = \sum_{d|n} f(d) F(n/d, \mathbf{x}/d),$$

where $\mathbf{x}/d = \langle x_1/d, ..., x_u/d \rangle$.

For $u = 0$, F is an arithmetical function ($F \in \mathcal{A} \equiv \mathcal{F}_0$) and operation (1.1) reduces to the Dirichlet convolution $*$ given by

$$(1.2) \qquad (f * F)(n) = \sum_{d|n} f(d) F(n/d).$$

More generally, if A is a regular convolution of Narkiewicz-type [5], see section 2, we define the operation $\circ_A$ by

$$(1.3) \qquad (f \circ_A F)(n, \mathbf{x}) = \sum_{d \in A(n)} f(d) F(n/d, \mathbf{x}/d),$$

which is a generalization of the A-convolution.

Operation (1.3) and its particular case (1.1) seem not to have appeared in the literature.

We show that operation $\circ_A$ has properties analogous to the generalized convolution

$$(f \circ F)(x) = \sum_{n \le x} f(n)F(x/n), \tag{1.4}$$

where $x \in \mathbf{R}, x \ge 1$, $f \in \mathcal{A}$ and $F : [1, \infty) \to \mathbf{C}$ is a function, see for ex. T. M. Apostol [1], pp. 39-40.

Then, using these properties we investigate certain Euler-type functions and Pillai-type functions and point out connections between them.

2 Regular convolutions

In this section we give the definition and some properties of the regular convolutions. Let $A(n)$ be a subset of the set $D(n)$ of positive divisors of n for each $n \in \mathbf{N}$. The A-convolution of the functions $f, g \in \mathcal{A}$ is given by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)g(n/d). \tag{2.1}$$

W.Narkiewicz [5] defined the A-convolution (2.1) to be regular if

(a) $\mathcal{A}$ is a commutative ring with unity δ (where $\delta(1) = 1$ and $\delta(n) = 0$ for all $n > 1$) with respect to ordinary addition and to $*_A$,

(b) the A-convolution of multiplicative functions is multiplicative,

(c) the function I, defined by $I(n) = 1$ for all $n \in \mathbf{N}$, has an inverse μ_A with respect to $*_A$ and $\mu_A(p^a) \in \{-1, 0\}$ for every prime power $p^a (a \ge 1)$.

It can be proved, see [5], that an A-convolution is regular if and only if

(i) $A(mn) = \{de : d \in A(m), e \in A(n)\}$ for every $m, n \in \mathbf{N}, (m, n) = 1$,

(ii) for every prime power $p^a (a \ge 1)$ there exists a divisor $t = t_A(p^a)$ of a, called the type of p^a with respect to A, such that $A(p^{it}) = \{1, p^t, p^{2t}, ..., p^{it}\}$ for every $i \in \{0, 1, ..., a/t\}$.

For example, the Dirichlet convolution D, where $D(n)$ is the set of all positive divisors of n, and the unitary convolution U, where $U(n)$ is the set of all unitary divisors of n (i.e. divisors d of n with $(d, n/d) = 1$), are regular.

The nonzero function $f \in \mathcal{A}$ is said to be A-multiplicative if $f(d)f(n/d) = f(n)$ for all $n, d \in \mathbf{N}, d \in A(n)$. Note that the D-multiplicative and U-multiplicative functions are the completely multiplicative and multiplicative functions, respectively. If f is A-multiplicative, then $f(g *_A h) = fg *_A fh$ for every $g, h \in \mathcal{A}$ and the inverse of f with respect to the A-convolution is $f^{-1} = \mu_A f$, cf. [11].

If S is an arbitrary subset of $\mathbf{N}$, let ρ_S stand for its characteristic function. We say that S is multiplicative if ρ_S is a multiplicative function. The generalized Möbius function $\mu_{S,A}$ is defined by

$$\mu_{S,A} *_A I = \rho_S. \tag{2.2}$$

For $k \in \mathbf{N}$, let $A_k(n) = \{d \in \mathbf{N} : d^k \in A(n^k)\}$. The A_k-convolution is regular whenever the A-convolution is regular, see [7], Theorem 3.1. Let $(a, b)_{A,k}$ denote the largest k-th power divisor of a which belongs to $A(b)$. Note that $(a, b)_{D,k} \equiv (a, b)_k$ is the greatest common k-th power divisor of a and b.

3 The operation $\circ_A$

Let A be a regular convolution and consider the operation $\circ_A$ defined by (1.3).

Theorem 3.1 *For every $f, g \in \mathcal{A}, F, G \in \mathcal{F}_u$,*

$$f \circ_A (F + G) = f \circ_A F + f \circ_A G, \tag{3.1}$$

$$(f + g) \circ_A F = f \circ_A F + g \circ_A F, \tag{3.2}$$

$$f \circ_A (g \circ_A F) = (f *_A g) \circ_A F, \tag{3.3}$$

$$\delta \circ_A F = F, \tag{3.4}$$

i.e. $\mathcal{F}_u$ is an $\mathcal{A}$-module.

Proof. (3.1), (3.2) and (3.4) yield at once by the definitions. We prove (3.3): For every $n \in \mathbf{N}, \mathbf{x} \in \mathbf{C}^u$:

$$(f \circ_A (g \circ_A F))(n, \mathbf{x}) = \sum_{d \in A(n)} f(d)(g \circ_A F)(n/d, \mathbf{x}/d) = \sum_{d \in A(n)} f(d) \sum_{e \in A(n/d)} g(e) F(n/(de), \mathbf{x}/(de)).$$

Denoting $de = \delta$, where $d \in A(n), e \in A(n/d)$ if and only if $\delta \in A(n), d \in A(\delta)$, cf. [7], Theorem 2.1, we have

$$(f \circ_A (g \circ_A F))(n, \mathbf{x}) = \sum_{\delta \in A(n)} \left(\sum_{d \in A(\delta)} f(d) g(\delta/d) \right) F(n/\delta, \mathbf{x}/\delta)$$

$$= \sum_{\delta \in A(n)} (f *_A g)(\delta) F(n/\delta, \mathbf{x}/\delta) = ((f *_A g) \circ_A F)(n, \mathbf{x}).$$

The next inversion formulae follow by Theorem 3.1.

Theorem 3.2 *(Inversion-formula) If $f \in \mathcal{A}, f(1) \neq 0$ and $F, G \in \mathcal{F}_u$, then the equation*

$$F(n, \mathbf{x}) = \sum_{d \in A(n)} f(d) G(n/d, \mathbf{x}/d)$$

implies

$$G(n, \mathbf{x}) = \sum_{d \in A(n)} f^{-1}(d) F(n/d, \mathbf{x}/d).$$

where f^{-1} is the inverse of f with respect to the A-convolution $*_A$.

Proof. If $F = f \circ_A G$, then $f^{-1} \circ_A F = f^{-1} \circ (f \circ_A G) = (f^{-1} *_A f) \circ G = \delta \circ_A G = G$ using Theorem 3.1.

Theorem 3.3 *(Möbius-type inversion) If $f \in \mathcal{A}$ is A-multiplicative and $F, G \in \mathcal{F}_u$, then the equation*

$$F(n, \mathbf{x}) = \sum_{d \in A(n)} f(d) G(n/d, \mathbf{x}/d)$$

implies and is implied by

$$G(n, \mathbf{x}) = \sum_{d \in A(n)} \mu_A(d) f(d) F(n/d, \mathbf{x}/d).$$

Proof. Apply Theorem 3.2 and the fact that $f^{-1} = \mu_A f$.

4 Applications to certain Euler-type and Pillai-type functions

Let A be a regular convolution, $k, u \in \mathbf{N}$ and let $F : \mathbf{N}^{u+1} \to \mathbf{C}$ a function of $u+1$ variables, whose values will be denoted by $F(n, \mathbf{a})$, where $\mathbf{a} = \langle a_1, a_2, ..., a_u \rangle \in \mathbf{N}^u$. Assume that $F(1, \mathbf{1}) \neq 0$, where $\mathbf{1} = \langle 1, 1, ..., 1 \rangle \in \mathbf{N}^u$, and that there exists a function $f_F = f \in \mathcal{A}$ with the property

$$F(n, d^k \mathbf{a}) = f(d) F(n/d, \mathbf{a}), \tag{4.1}$$

for every $n, d \in \mathbf{N}, d \in A_k(n)$ and $\mathbf{a} \in \mathbf{N}^u$, where $d^k \mathbf{a} = \langle d^k a_1, d^k a_2, ..., d^k a_u \rangle$.

Remark. The function f satisfying (4.1) is A_k-multiplicative:

$$f(de) = \frac{F(de, (de)^k \mathbf{a})}{F(1, \mathbf{a})} = \frac{f(d) F(e, e^k \mathbf{a})}{F(1, \mathbf{a})} = \frac{f(d) f(e) F(1, \mathbf{a})}{F(1, \mathbf{a})} = f(d) f(e)$$

for every $d, e \in \mathbf{N}$, $d \in A_k(de)$ and for $\mathbf{a} = \langle 1, 1, ..., 1 \rangle$.

If $\mathbf{x} = \langle x_1, x_2, ..., x_u \rangle, \mathbf{y} = \langle y_1, y_2, ..., y_u \rangle \in \mathbf{R}^u$, we write $\mathbf{x} \leq \mathbf{y}$ for $x_i \leq y_i, i \in \{1, 2, ..., u\}$. For an arbitrary subset S of $\mathbf{N}$ and for $\mathbf{1} \leq \mathbf{x}$ define the functions belonging to $\mathcal{F}_u$:

$$\phi^{(u)}_{S,A,k,F}(n, \mathbf{x}) = \sum_{\substack{\mathbf{1} \leq \mathbf{a} \leq \mathbf{x}^k \\ (((\mathbf{a}), n^k)_{A,k})^{1/k} \in S}} F(n, \mathbf{a}), \tag{4.2}$$

where $\mathbf{x}^k = \langle x_1^k, x_2^k, ..., x_u^k \rangle$ and $(\mathbf{a})$ is the gcd $(a_1, a_2, ..., a_u)$,

$$P^{(u)}_{k,F}(n, \mathbf{x}) = \sum_{\mathbf{1} \leq \mathbf{a} \leq \mathbf{x}^k} F(n, \mathbf{a}). \tag{4.3}$$

Theorem 4.1 *If $S \subseteq \mathbf{N}$, A is a regular convolution and F is defined by (4.1), then*

$$\phi^{(u)}_{S,A,k,F} = \mu_{S,A_k} f \circ_{A_k} P^{(u)}_{k,F}. \tag{4.4}$$

Proof. Using that $d^k \in A((a, b)_{A,k})$ if and only if $d^k | a$ and $d^k \in A(b)$, see [7], Theorem 4.2, we have by (2.2),

$$\begin{aligned}
\phi^{(u)}_{S,A,k,F}(n, \mathbf{x}) &= \sum_{\mathbf{1} \leq \mathbf{a} \leq \mathbf{x}^k} F(n, \mathbf{a}) \rho_S((((\mathbf{a}), n^k)_{A,k})^{1/k}) \\
&= \sum_{\mathbf{1} \leq \mathbf{a} \leq \mathbf{x}^k} F(n, \mathbf{a}) \sum_{\substack{d^k \in A(n^k) \\ d^k | (\mathbf{a})}} \mu_{S,A_k}(d) \\
&= \sum_{d \in A_k(n)} \mu_{S,A_k}(d) \sum_{\substack{\mathbf{1} \leq \mathbf{a} \leq \mathbf{x}^k \\ d^k | a_1, d^k | a_2, ..., d^k | a_u}} F(n, \mathbf{a}),
\end{aligned}$$

where denoting $a_i = d^k b_i, i \in \{1, 2, ..., u\}$ we get

$$\phi^{(u)}_{S,A,k,F}(n, \mathbf{x}) = \sum_{d \in A_k(n)} \mu_{S,A_k}(d) \sum_{\mathbf{1} \leq \mathbf{b} \leq (\mathbf{x}/d)^k} F(n, d^k \mathbf{b})$$

$$= \sum_{d \in A_k(n)} \mu_{S,A_k}(d) f(d) \sum_{\mathbf{1} \leq \mathbf{b} \leq (\mathbf{x}/d)^k} F(n/d, \mathbf{b})$$

$$= \sum_{d \in A_k(n)} \mu_{S,A_k}(d) f(d) P^{(u)}_{k,F}(n/d, \mathbf{x}/d)$$

$$= (\mu_{S,A_k} f \circ_{A_k} P^{(u)}_{k,F})(n, \mathbf{x}).$$

Theorem 4.2 *If $S \subseteq \mathbf{N}$, A is a regular convolution and F is defined by (4.1), then*

$$P^{(u)}_{k,F} = h \circ_{A_k} \phi^{(u)}_{S,A,k,F}, \tag{4.5}$$

where $h = f *_{A_k} (f\rho_S)^{-1}$. If $S = \{1\}$, then $h = f$.

Proof. Applying Theorem 3.3 we get from Theorem 4.1 relation (4.5) with $h = (\mu_{S,A_k} f)^{-1}$ which is, using (2.2), $h = ((\mu_{A_k} *_{A_k} \rho_S) f)^{-1} = (\mu_{A_k} f *_{A_k} \rho_S f)^{-1} = (\mu_{A_k} f)^{-1} *_{A_k} (\rho_S f)^{-1} = f *_{A_k} (f\rho_S)^{-1}$. If $S = \{1\}$, then $\rho_S = \delta$ and $h = f$.

If $F(n, \mathbf{a}) = 1$ for every $\langle n, \mathbf{a} \rangle \in N^{u+1}$, then $P^{(u)}_{k,F}(n, \mathbf{x}) = [x_1^k][x_2^k]...[x_u^k]$ and $\phi^{(u)}_{S,A,k,F}(n, \mathbf{x}) \equiv \phi^{(u)}_{S,A,k}(n, \mathbf{x})$ is the Legendre-type function introduced by P. Haukkanen [3] which is a common generalization of a large number of Euler-type functions, and Theorem 4.1. reduces to Theorem 1 of [3]:

$$\phi^{(u)}_{S,A,k}(n, \mathbf{x}) = \sum_{d \in A_k(n)} \mu_{S,A_k}(d) [(x_1/d)^k][(x_2/d)^k]...[(x_u/d)^k], \tag{4.6}$$

see also L. Tóth and P. Haukkanen [9].

For $F(n, \mathbf{a}) = \exp(2\pi i(m_1 a_1 + ... + m_u a_u)/n^k)$ and $f_F(n) = 1$ we obtain the generalization of the Ramanujan sum, cf. P. Haukkanen [3].

Next we consider the following particular case of the functions $\phi^{(u)}_{S,A,k,F}(n, \mathbf{x})$ and $P^{(u)}_{k,F}(n, \mathbf{x})$ defined by (4.2) and (4.3). Let g be an A_k-multiplicative function and denote

$$g_{A,k}(n, \mathbf{a}) = g((((\mathbf{a}), n^k)_{A,k})^{1/k}).$$

Now for $F = g_{A,k}$, we have $f_F = g$ and let $\phi^{(u)}_{S,A,k,g_{A,k}} \equiv \phi^{(u)}_{S,A,k,g}, P^{(u)}_{k,g_{A,k}} \equiv P^{(u)}_{A,k,g}$, where

$$\phi^{(u)}_{S,A,k,g}(n, \mathbf{x}) = \sum_{\substack{\mathbf{1} \leq \mathbf{a} \leq \mathbf{x}^k \\ (((\mathbf{a}), n^k)_{A,k})^{1/k} \in S}} g((((\mathbf{a}), n^k)_{A,k})^{1/k}), \tag{4.7}$$

$$P^{(u)}_{A,k,g}(n, \mathbf{x}) = \sum_{\mathbf{1} \leq \mathbf{a} \leq \mathbf{x}^k} g((((\mathbf{a}), n^k)_{A,k})^{1/k}). \tag{4.8}$$

Here $P^{(u)}_{A,k,g}(n, \mathbf{x})$ is a generalization of the Pillai [6] function $P(n) = \sum_{i=1}^n (i, n)$ which is obtained for $A = D, k = u = 1, x_1 = n$ and $g(n) = n, n \in \mathbf{N}$.

Function $\phi^{(u)}_{S,A,k,g}(n, \mathbf{x})$ is a generalization of the Euler-type function $\phi^{(u)}_{\{1\},A,k}(n, \mathbf{x})$ of above which is obtained for $S = \{1\}$ and for an arbitrary A_k-multiplicative function g.

From Theorems 4.1. and 4.2. we immediately have

Corollary 4.1 *If $S \subseteq \mathbf{N}$ and A is a regular convolution, then*

$$\phi^{(u)}_{S,A,k,g} = \mu_{S,A_k} g \circ_{A_k} P^{(u)}_{A,k,g}. \tag{4.9}$$

Corollary 4.2 *If $S \subseteq \mathbf{N}$ and A is a regular convolution, then*

$$P^{(u)}_{A,k,g} = h \circ_{A_k} \phi^{(u)}_{S,A,k,g}, \tag{4.10}$$

where $h = g *_{A_k} (g\rho_S)^{-1}$. If $S = \{1\}$, then $h = g$.

From Corollary 4.2. we obtain

$$P^{(u)}_{A,k,g}(n, \mathbf{x}) = \sum_{d \in A_k(n)} h(d) \phi^{(u)}_{S,A,k,g}(n/d, \mathbf{x}/d). \tag{4.11}$$

We obtain the following Legendre-type evaluations for $\phi^{(u)}_{S,A,k,g}(n, \mathbf{x})$ and $P^{(u)}_{A,k,g}(n, \mathbf{x})$ given by (4.7) and (4.8), respectively.

Theorem 4.3 *If $S \subseteq \mathbf{N}$, A is a regular convolution, $k, u, n \in \mathbf{N}, \mathbf{1} \leq \mathbf{x} \in \mathbf{R}^u$ and g is an A_k-multiplicative function, then*

$$\phi^{(u)}_{S,A,k,g}(n, \mathbf{x}) = \sum_{d \in A_k(n)} (\rho_S g *_{A_k} \mu_{A_k})(d)[(x_1/d)^k][(x_2/d)^k]...[(x_u/d)^k], \tag{4.12}$$

$$P^{(u)}_{A,k,g}(n, \mathbf{x}) = \sum_{d \in A_k(n)} (g *_{A_k} \mu_{A_k})(d)[(x_1/d)^k][(x_2/d)^k]...[(x_u/d)^k]. \tag{4.13}$$

Proof. Combining (4.11) and (4.6) for $S = \{1\}$ and using property (3.3) of the operation $\circ_A$ we obtain (4.13). Now (4.12) is a consequence of (4.9), (4.13) and of the fact that $\mu_{S,A_k} g *_{A_k} g = \rho_S g$, see (2.2).

The following direct proof shows that (4.13) is valid for an arbitrary arithmetical function g:

Noting that $g = g *_{A_k} \mu_{A_k} *_{A_k} I$ we can write

$$P^{(u)}_{A,k,g}(n, \mathbf{x}) = \sum_{\mathbf{1} \leq \mathbf{a} \leq \mathbf{x}^k} \sum_{\substack{d \in A_k(n) \\ d^k | (\mathbf{a})}} (g *_{A_k} \mu_{A_k})(d)$$

$$= \sum_{d \in A_k(n)} (g *_{A_k} \mu_{A_k})(d) \sum_{\substack{\mathbf{1} \leq \mathbf{a} \leq \mathbf{x}^k \\ d^k | (\mathbf{a})}} 1$$

$$= \sum_{d \in A_k(n)} (g *_{A_k} \mu_{A_k})(d)[(x_1/d)^k][(x_2/d)^k]...[(x_u/d)^k],$$

which is the desired result.

For other generalizations and properties of Pillai-type functions, including arithmetical evaluations and asymptotic formulae we refer to [4] and [10].

A further interesting particular case is $F(n, \mathbf{a}) = F(\mathbf{a})$ depending only on $\mathbf{a}$, where F is a completely multiplicative function of u variables (i.e. $F(a_1b_1, a_2b_2, \ldots, a_ub_u) = F(\mathbf{a})F(\mathbf{b})$

for all $\mathbf{a}, \mathbf{b} \in \mathbf{N}^u$). Then $f(n) = F(n, n, \dots, n)^k$, which leads to another generalizations of the Euler function. For example, if $F(\mathbf{a}) = E_s(\mathbf{a}) = (a_1 a_2 \cdots a_u)^s$, then

$$\phi^{(u)}_{S,A,k,E_s}(n, \mathbf{x}) \equiv \phi^{(u)}_{S,A,k,s}(n, \mathbf{x}) = \sum_{\substack{\mathbf{1} \le \mathbf{a} \le \mathbf{x}^k \\ (((\mathbf{a}), n^k)_{A,k})^{1/k} \in S}} (a_1 a_2 \cdots a_u)^s,$$

$$P^{(u)}_{k,E_s}(n, \mathbf{x}) \equiv P^{(u)}_{k,s}(n, \mathbf{x}) = \sum_{\mathbf{1} \le \mathbf{a} \le \mathbf{x}^k} (a_1 a_2 \cdots a_u)^s.$$

Special cases of $\phi^{(u)}_{S,A,k,E_s}(n, x)$ were investigated by several authors, for ex. H. Davenport [2] in case $A = D, S = \{1\}, k = u = 1, x = n$.

Corollary 4.3 *If F is a completely multiplicative function of u variables, then*

$$\phi^{(u)}_{S,A,k,F} = \mu_{S,A_k} F((\cdot), (\cdot), \dots, (\cdot))^k \circ_{A_k} P^{(u)}_{k,F}.$$

In particular,

$$\phi^{(1)}_{S,A,k,F} = \mu_{S,A_k} F^k \circ_{A_k} P^{(1)}_{k,F},$$

$$\phi^{(1)}_{S,A,k,s}(n, x) = \sum_{d \in A_k(n)} \mu_{S,A_k}(d) d^{ks} \sum_{e \le (x/d)^k} e^s.$$

For $u = 1, s = 0, x = n^k$ let $\phi^{(1)}_{S,A,k,0}(n, n^k) \equiv \phi_{S,A,k}(n)$, where

$$\phi_{S,A,k} = \mu_{S,A_k} *_{A_k} E_k. \tag{4.14}$$

Corollary 4.4 ($s = 1, x = n^k$) *For every $n \in \mathbf{N}$,*

$$\sum_{\substack{1 \le a \le n^k \\ ((a, n^k)_{A,k})^{1/k} \in S}} a = \frac{n^k}{2} (\phi_{S,A,k}(n) + \rho_S(n)).$$

Proof. Applying Corollary 4.3.,

$$\sum_{\substack{1 \le a \le n^k \\ ((a, n^k)_{A,k})^{1/k} \in S}} a = \sum_{d \in A_k(n)} \mu_{S,A_k}(d) d^k \frac{n^k}{2d^k} \left(\frac{n^k}{d^k} + 1 \right)$$

$$= \frac{n^k}{2} \left(\sum_{d \in A_k(n)} \mu_{S,A_k}(d) \frac{n^k}{d^k} + \sum_{d \in A_k(n)} \mu_{S,A_k}(d) \right) = \frac{n^k}{2} (\phi_{S,A,k}(n) + \rho_S(n)),$$

by (4.14) and (2.2).

For $A = D, k = 1$ we reobtain Lemma 3 of [8].

For $S = \{1\}$ let $\phi_{\{1\},A,k}(n) \equiv \phi_{A,k}(n)$, the Euler-type function due to V. Sita Ramaiah [7].

Corollary 4.5 ($S = \{1\}$) *For every $n \in \mathbf{N}, n > 1$,*

$$\sum_{\substack{1 \le a \le n^k \\ (a, n^k)_{A,k} = 1}} a = \frac{n^k \phi_{A,k}(n)}{2}.$$

This is well-known in case $A = D, k = 1$. From Corollary 4.3 we obtain by similar arguments:

Corollary 4.6 $(s = 2, x = n^k)$ *For every* $n \in \mathbf{N}$,

$$\sum_{\substack{1\le a\le n^k \\ ((a,n^k)_{A,k})^{1/k}\in S}} a^2 = \frac{n^{2k}}{3}\phi_{S,A,k}(n) + \frac{n^{2k}}{2}\rho_S(n) + \frac{(-1)^{\omega(n)}}{6}\phi_{S,A,k}(n)\prod_{p^a\|n} p^{tk},$$

where $t = t_{A_k}(p^a)$ *is the type of* p^a *with respect to* A_k.

Corollary 4.7 $(S = \{1\})$ *For every* $n \in \mathbf{N}, n > 1$,

$$\sum_{\substack{1\le a\le n^k \\ ((a,n^k)_{A,k}=1}} a^2 = \frac{n^{2k}\phi_{A,k}(n)}{3} + \frac{(-1)^{\omega(n)}}{6}\phi_{A,k}(n)\prod_{p^a\|n} p^{tk}.$$

This result is well known for $A = D, k = 1$. The next asymptotic formula is a generalization of a result due to H. Davenport [2] in the case mentioned above.

Theorem 4.4 *If* $S \subseteq \mathbf{N}$, A *is an arbitrary regular convolution and* $s \ge 0$, *then*

$$\phi^{(1)}_{S,A,k,s}(n,x) = \frac{\phi_{S,A,k}(n)x^{k(s+1)}}{n^k(s+1)} + O(f_S(n)x^{ks}),$$

where $f_S(n) = \sum_{d\in A_k(n)} 1 \equiv \tau_{A_k}(n)$ *if* S *is multiplicative and* $f_S(n) = \sum_{d\in A_k(n)} \tau_{A_k}(d)$ *otherwise.*

Proof. Using the familiar estimate

$$\sum_{n\le x} n^s = \frac{x^{s+1}}{s+1} + O(x^s),$$

valid for $s \ge 0$ we obtain from Corollary 4.3.

$$\phi_{S,A,k,s}(n,x) = \sum_{d\in A_k(n)} d^{ks}\mu_{S,A_k}(d)\sum_{e\le(x/d)^k} e^s$$

$$= \sum_{d\in A_k(n)} d^{ks}\mu_{S,A_k}(d)\left(\frac{1}{s+1}(x/d)^{k(s+1)} + O((x/d)^{ks})\right)$$

$$= \frac{x^{k(s+1)}}{s+1}\sum_{d\in A_k(n)} \mu_{S,A_k}(d)/d^k + O\left(x^{ks}\sum_{d\in A_k(n)} |\mu_{S,A_k}(d)|\right)$$

$$= \frac{\phi_{S,A,k}(n)x^{k(s+1)}}{n^k(s+1)} + O(f_S(n)x^{ks}),$$

using (4.14) and that $|\mu_{S,A_k}(n)| \le 1$ if S is multiplicative and $|\mu_{S,A_k}(n)| \le \tau_{A_k}(n)$ for an arbitrary $S \subseteq \mathbf{N}$, see Lemmas 1 and 2 of [9].

Our final formula is regarding the Pillai-type function (4.8).

Theorem 4.5 *If A is an arbitrary regular convolution and g is an arbitrary arithmetical function, then*

$$P^{(1)}_{k,g}(n,x) \equiv \sum_{1\le a\le x^k} g(((a,n^k)_{A,k})^{1/k}) = \frac{(g *_{A_k} \phi_{A,k})(n)}{n^k} x^k + O((|g *_{A_k} \mu_{A_k}| *_{A_k} I)(n)).$$

Proof. From (4.13) we obtain

$$P^{(1)}_{A,k,g}(n,x) = \sum_{d\in A_k(n)} (g *_{A_k} \mu_{A_k})(d)[(x/d)^k] = \sum_{d\in A_k(n)} (g *_{A_k} \mu_{A_k})(d) \left((x/d)^k + O(1)\right)$$

$$= (x/n)^k \sum_{d\in A_k(n)} (g *_{A_k} \mu_{A_k})(d)(n/d)^k + O\left(\sum_{d\in A_k(n)} |(g *_{A_k} \mu_{A_k})(d)|\right)$$

and we use (4.14).

For different choices of g we obtain from Theorem 4.5 various formulae. We give here only the following particular case.

Corollary 4.8 ($g = \tau_A, k = 1$)

$$\sum_{1\le a\le x} \tau_A((a,n)_A) = \frac{\sigma_A(n)}{n} x + O(\tau_A(n)),$$

where $\sigma_A(n) = \sum_{d\in A(n)} d$.

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