

A GENERALIZATION OF EULER'S ϕ -FUNCTION WITH RESPECT TO A SET OF POLYNOMIALS

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Abstract. P. Kesava Menon's generalization $\phi_f(n)$ of Euler's ϕ -function is defined as the number of integers $x \pmod{n}$ such that $(f(x), n) = 1$, where f is a polynomial with integral coefficients. In this paper we define a generalization of Euler's ϕ -function, which combines $\phi_f(n)$ and a large number of other generalizations of Euler's ϕ -function. We give an arithmetical evaluation and an asymptotic formula for our new generalization of Euler's ϕ -function.

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1. INTRODUCTION

The Euler function $\phi(n)$ is defined as the number of integers $x \pmod{n}$ such that $(x, n) = 1$. The well-known Jordan function $J_u(n)$ is a generalized Euler function defined as the number of ordered u -tuples $\langle x_1, x_2, \dots, x_u \rangle \pmod{n}$ such that $((x_j), n) = 1$, where $(x_j) = (x_1, x_2, \dots, x_u)$, the gcd of x_1, x_2, \dots, x_u . E. Cohen [5], [6] defines the generalized Euler function $\phi_k(n)$ as the number of integers $x \pmod{n^k}$ such that $(x, n^k)_k = 1$, where $(a, b)_k$ denotes the greatest common k -th power divisor of a and b . For an arbitrary set S of positive integers E. Cohen [7] defines the generalized Euler function $\phi_S(n)$ as the number of integers $x \pmod{n}$ such that $(x, n) \in S$. P. J. McCarthy [12] involves Narkiewicz's [14] convolution in generalized Euler functions.

In [9] the second author of the present paper combines the above generalizations of Euler's function.

The following generalization of $\phi(n)$ is due to P. Kesava Menon [11], see also H. Stevens [17] and J. Chidambaraswamy [2], [3]. For a polynomial f with integral coefficients let $\phi_f(n)$ be the number of integers $x \pmod{n}$ such that $(f(x), n) = 1$. P. G. Garcia and S. Ligh [8] introduce another generalization of $\phi(n)$, namely for an arithmetic progression $D(s, d, n) = \{s, s + d, s + 2d, \dots, s + (n - 1)d\}$, where $(s, d) = 1$, let $\phi(s, d, n)$ denote the number of elements x in $D(s, d, n)$ such that $(x, n) = 1$. Observe that $\phi(s, d, n)$ is a special case of the function $\phi_f(n)$ for $f(x) = s + (x - 1)d$.

In this paper we combine the generalization of [9] and the function $\phi_f(n)$ as follows.

Let A be a mapping from the set \mathbf{N} of positive integers to the set of subsets of \mathbf{N} such that for each $n \in \mathbf{N}$, $A(n)$ consists entirely of divisors of n . The A -convolution of arithmetical functions f and g is defined as

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)g(n/d).$$

W. Narkiewicz [14] defined an A -convolution to be regular if

(a) the set of arithmetical functions is a commutative ring with unity with respect to ordinary addition and the A -convolution,

(b) the A -convolution of multiplicative functions is multiplicative,

(c) the function E , defined by $E(n) = 1$ for all $n \in \mathbf{N}$, has an inverse μ_A with respect to the A -convolution and $\mu(p^a) \in \{-1, 0\}$ for every prime power p^a .

In this paper we consider regular A -convolutions, see also [13, Chapter 4], [16]. For example, the Dirichlet convolution D , where $D(n) = \{d \in \mathbf{N} : d|n\}$, and the unitary convolution U , where $U(n) = \{d \in \mathbf{N} : d|n, (d, n/d) = 1\}$, are regular.

For $k \in \mathbf{N}$, let $A_k(n) = \{d \in \mathbf{N} : d^k \in A(n^k)\}$. It is well-known that the A_k -convolution is regular whenever the A -convolution is regular. The symbol $(a, b)_{A,k}$ denotes the largest k -th power divisor of a which belongs to $A(b)$. Note that $(a, b)_{D,k} = (a, b)_k$.

Now let $\mathbf{n} = \langle n_1, n_2, \dots, n_u \rangle$ be an ordered u -tuple of positive integers and let k be a positive integer. We say that the u -tuples $\langle x_1, x_2, \dots, x_u \rangle$ and $\langle y_1, y_2, \dots, y_u \rangle$ are congruent $(\text{mod } \mathbf{n}^k)$ if $x_i \equiv y_i \pmod{n_i^k}$ for every $i = 1, 2, \dots, u$. Let $F = \{f_1, f_2, \dots, f_u\}$ be a set of polynomials with integral coefficients, S be an arbitrary set of positive integers, A be a regular convolution and $n|(n_1, n_2, \dots, n_u)$. We define the generalized Euler function $\phi_{F,S,A,k}(\mathbf{n}, n)$ as the number of incongruent u -tuples $\langle x_1, x_2, \dots, x_u \rangle \pmod{\mathbf{n}^k}$ such that $((f_j(x_j), n^k)_{A,k})^{1/k} \in S$. We give an arithmetical evaluation and an asymptotic formula for our new generalization of Euler's function. In the asymptotic formula we confine ourselves to a special case of Narkiewicz's regular convolution, called *cross-convolution*, including the Dirichlet convolution and the unitary convolution. The method we use here is elementary, it is described in detail and applied for various types of arithmetical functions in [21] and [22].

For special cases of our results we refer to the papers given in the bibliography and to the book of P. J. McCarthy [13].

2. PRELIMINARIES

If A is a regular convolution, then for every prime power $p^a (a \geq 1)$ there exists a positive integer $t = t_A(p^a)$, called the type of p^a with respect to A , such that $A(p^a) = \{1, p^t, p^{2t}, \dots, p^{st}\}$, $st = a$ and $p^t \in A(p^{2t}), p^{2t} \in A(p^{3t}), \dots, p^{(s-1)t} \in A(p^a)$.

A positive integer n is said to be A -primitive if $A(n) = \{1, n\}$. It follows that the Möbius-type function μ_A is multiplicative and for all prime powers $p^a (a \geq 1)$,

$$(1) \quad \mu_A(p^a) = \begin{cases} -1, & \text{if } p^a \text{ is } A\text{-primitive (i.e. } t_A(p^a) = a), \\ 0, & \text{otherwise.} \end{cases}$$

Let S be a subset of \mathbf{N} and let ρ_S denote the characteristic function of the set S , that is $\rho_S(n) = 1$ if $n \in S$, and $\rho_S(n) = 0$ if $n \notin S$. The generalized Möbius function $\mu_{S,A}$ is defined by

$$(2) \quad \mu_{S,A} *_{A} E = \rho_S,$$

where $E(n) = 1$ for all $n \in \mathbf{N}$. If $S = \{1\}$, then $\mu_{S,A} = \mu_A$, and if $A = D$, then $\mu_A = \mu$, the classical Möbius function. For further special cases of $\mu_{S,A}$ we refer to [9].

We say that S is multiplicative if its characteristic function ρ_S is multiplicative, i.e. $1 \in S$ and $mn \in S$ if, and only if, $m \in S, n \in S$ for every $m, n \in \mathbf{N}$ with $(m, n) = 1$.

Lemma 1. *The function $\mu_{S,A}$ is multiplicative if and only if S is multiplicative, and in this case*

$$\mu_{S,A}(n) = \prod_{p^a || n} (\rho_S(p^a) - \rho_S(p^{a-t})),$$

for every $n \in \mathbf{N}$, where $t = t_A(p^a)$ is the type of p^a with respect to A and $p^a || n$ means $p^a | n$ and $p^{a+1} \nmid n$. If S is multiplicative, then $\mu_{S,A}(n) \in \{-1, 0, 1\}$ for every $n \in \mathbf{N}$.

Proof. This is an immediately consequence of (2) and (1).

Remark 1. In particular, if S is multiplicative, then

$$\mu_{S,D}(n) = \prod_{p^a || n} (\rho_S(p^a) - \rho_S(p^{a-1})),$$

$$\mu_{S,U}(n) = \prod_{p^a || n} (\rho_S(p^a) - 1),$$

for every $n \in \mathbf{N}$.

Lemma 2. *For every subset S and for every regular convolution A we have $|\mu_{S,A}(n)| \leq \tau(n)$ for every $n \in \mathbf{N}$, where $\tau(n)$ stands for the number of divisors of n , and $\mu_{S,A}(n) = O(n^\varepsilon)$ for every $\varepsilon > 0$.*

Proof. By (2) and by Möbius inversion

$$|\mu_{S,A}(n)| = \left| \sum_{d \in A(n)} \rho_S(d) \mu_A(n/d) \right| \leq \sum_{d \in A(n)} |\mu_A(n/d)| = 2^{\omega(n)} \leq \tau(n) = O(n^\varepsilon)$$

for every $\varepsilon > 0$, where $\omega(n)$ denotes the number of distinct prime factors of n .

3. ARITHMETICAL EVALUATIONS

For a polynomial f with integral coefficients let $N_f(n)$ denote the number of incongruent solutions (mod n) of the congruence $f(x) \equiv 0 \pmod{n}$. It is well-known that the function N_f is multiplicative. Define the function N_F by $N_F(n) = N_{f_1}(n)N_{f_2}(n)\dots N_{f_u}(n)$ for each $n \in \mathbf{N}$. It follows that the function N_F is multiplicative.

Theorem 1. For every F, S, A, k, \mathbf{n} and n with $n|(n_1, n_2, \dots, n_u)$ we have

$$\phi_{F,S,A,k}(\mathbf{n}, n) = (n_1 n_2 \dots n_u)^k \sum_{e \in A_k(n)} \mu_{S,A_k}(e) e^{-ku} N_F(e^k).$$

Proof. By the definition of $\phi_{F,S,A,k}(\mathbf{n}, n)$, by (2) and using that $d^k \in A((a, b)_{A,k})$ if and only if $d^k|a$ and $d^k \in A(b)$, see [16, Theorem 4.2], we get

$$\begin{aligned} \phi_{F,S,A,k}(\mathbf{n}, n) &= \sum_{x_1 \pmod{n_1^k}} \sum_{x_2 \pmod{n_2^k}} \dots \sum_{x_u \pmod{n_u^k}} \rho_S(((f_j(x_j), n^k)_{A,k})^{1/k}) \\ &= \sum_{x_1 \pmod{n_1^k}} \sum_{x_2 \pmod{n_2^k}} \dots \sum_{x_u \pmod{n_u^k}} \sum_{\substack{e \in A_k(n) \\ e^k | f_j(x_j) \\ j=1,2,\dots,u}} \mu_{S,A_k}(e) \\ &= \sum_{e \in A_k(n)} \mu_{S,A_k}(e) \sum_{\substack{x_1 \pmod{n_1^k} \\ e^k | f_1(x_1)}} \sum_{\substack{x_2 \pmod{n_2^k} \\ e^k | f_2(x_2)}} \dots \sum_{\substack{x_u \pmod{n_u^k} \\ e^k | f_u(x_u)}} 1. \end{aligned}$$

Here for each $i = 1, 2, \dots, u$ the number of incongruent solutions $(\text{mod } n_i^k)$ of the congruence $f_i(x) \equiv 0 \pmod{e^k}$ is $N_{f_i}(e^k)(n_i/e)^k$. Thus

$$\phi_{F,S,A,k}(\mathbf{n}, n) = \sum_{e \in A_k(n)} \mu_{S,A_k}(e) N_{f_1}(e^k)(n_1/e)^k N_{f_2}(e^k)(n_2/e)^k \dots N_{f_u}(e^k)(n_u/e)^k,$$

which completes the proof.

Theorem 2. If S is multiplicative, then

$$\phi_{F,S,A,k}(\mathbf{n}, n) = (n_1 n_2 \dots n_u)^k \prod_{p^a | n} \left(1 + \sum_{i=1}^{a/t} (\rho_S(p^{it}) - \rho_S(p^{(i-1)t})) p^{-itku} N_F(p^{itk}) \right),$$

where $t = t_{A_k}(p^a)$ is the type of p^a with respect to A_k .

Proof. Theorem 2 is a direct consequence of Theorem 1 and Lemma 1.

Corollary 1. If $S = \{1\}$, then

$$\phi_{F,S,A,k}(\mathbf{n}, n) = (n_1 n_2 \dots n_u)^k \prod_{p^a | n} (1 - N_F(p^{tk}) p^{-tku}),$$

where $t = t_{A_k}(p^a)$.

If $f_i(x) = s_i + (x-1)d_i^k$, $i = 1, 2, \dots, u$, then let $\phi_{F,S,A,k}(\mathbf{n}, n) = \phi_{S,A,k}(\mathbf{s}, \mathbf{d}, \mathbf{n}, n)$, which is the number of ordered u -tuples $\langle x_1, x_2, \dots, x_u \rangle$ in $D_k(\mathbf{s}, \mathbf{d}, \mathbf{n})$ such that $((x_j, n^k)_{A,k})^{1/k} \in S$, where $D_k(\mathbf{s}, \mathbf{d}, \mathbf{n}) = D_k(s_1, d_1, n_1) \times D_k(s_2, d_2, n_2) \times \dots \times D_k(s_u, d_u, n_u)$ and $D_k(s_i, d_i, n_i) = \{s_i, s_i + d_i^k, s_i + 2d_i^k, \dots, s_i + (n_i^k - 1)d_i^k\}$.

This function is a direct generalization of the function $\phi(s, d, n)$ of P. G. Garcia and S. Ligh [8] and of the functions investigated by the first author of the present paper in [20].

Taking into account that in this case $N_{f_i}(n) = (d_i^k, n)$ if $(d_i^k, n)|s_i$ and $N_{f_i}(n) = 0$ otherwise, from Theorem 1 we get the following

Corollary 2. For every $S, A, k, \mathbf{s}, \mathbf{d}, \mathbf{n}$ and n with $n|(n_1, n_2, \dots, n_u)$ we have

$$\phi_{S,A,k}(\mathbf{s}, \mathbf{d}, \mathbf{n}, n) = (n_1 n_2 \dots n_u)^k \sum_{\substack{e \in A_k(n) \\ (e, d_i)^k | s_i \\ i=1,2,\dots,u}} \mu_{S,A_k}(e) e^{-ku} (e, d_1)^k (e, d_2)^k \dots (e, d_u)^k.$$

Corollary 3. If $(s_i, d_i^k)_k = 1$, $i = 1, 2, \dots, u$, then

$$\phi_{S,A,k}(\mathbf{s}, \mathbf{d}, \mathbf{n}, n) = (n_1 n_2 \dots n_u)^k \sum_{\substack{e \in A_k(n) \\ (e, d_i)=1 \\ i=1,2,\dots,u}} \mu_{S,A_k}(e) e^{-ku}.$$

Proof. Since $(s_i, d_i^k)_k = 1$, we have $(e, d_i)^k | s_i$ if and only if $(e, d_i) = 1$.

Corollary 4. If $S = \{1\}$ and $(s_i, d_i^k)_k = 1$, $i = 1, 2, \dots, u$, then

$$\phi_{S,A,k}(\mathbf{s}, \mathbf{d}, \mathbf{n}, n) = (n_1 n_2 \dots n_u)^k \prod_{\substack{p^t \in A_k(n) \\ p|d}} (1 - p^{-tku}),$$

where $d = d_1 d_2 \dots d_u$ and the product is over the A_k -primitive prime powers p^t such that $p^t \in A_k(n)$ and $p \nmid d$.

Remark 2. It should be noted that we do not need the assumption $(s_i, d_i^k)_k = 1$, $i = 1, 2, \dots, u$, in defining the function $\phi_{S,A,k}(\mathbf{s}, \mathbf{d}, \mathbf{n}, n)$. This kind of assumption is made in the previous arithmetical evaluations of generalized Euler functions in arithmetic progressions, see [8], [20].

Now for an arbitrary set F of polynomials and for $n_1 = n_2 = \dots = n_u = n$, let $\phi_{F,S,A,k}(\mathbf{n}, n) = \phi_{F,S,A,k}(n)$.

Theorem 3. If S is multiplicative, then the arithmetical function $\phi_{F,S,A,k}(n)$ is multiplicative.

Proof. It has been noted that $N_F(n)$ is multiplicative. If S is multiplicative, then μ_{S,A_k} is also multiplicative by Lemma 1. Hence, according to Theorem 1, $\phi_{F,S,A,k}(n)$ is the A_k -convolution of two multiplicative functions and it is multiplicative too.

4. ASYMPTOTIC FORMULAE

We need the following well-known estimates.

Lemma 3.

$$(3) \quad \sum_{n \leq x} n^{-s} = O(x^{1-s}), \quad 0 < s < 1,$$

$$(4) \quad \sum_{n \leq x} n^{-s} = \begin{cases} O(1), & s > 1, \\ O(\log x), & s = 1, \end{cases}$$

$$(5) \quad \sum_{n > x} n^{-s} = O(x^{1-s}), \quad s > 1.$$

Lemma 4. (see [19, Lemma 5 and Lemma 8])

$$(6) \quad \sum_{n \leq x} \frac{\tau(n)}{n^s} = \begin{cases} O(x^{1-s} \log x), & 0 < s < 1, \\ O(\log^2 x), & s = 1, \\ O(1), & s > 1, \end{cases}$$

$$(7) \quad \sum_{n \leq x} \frac{\tau^2(n)}{n^s} = \begin{cases} O(x^{1-s} \log^3 x), & 0 < s < 1, \\ O(\log^4 x), & s = 1, \\ O(1), & s > 1, \end{cases}$$

$$(8) \quad \sum_{n > x} \frac{\tau(n)}{n^s} = O(x^{1-s} \log x), \quad s > 1.$$

Lemma 5. (see [1, Lemma 2.3]) *If $s \geq 0$ and $a \in \mathbf{N}$, then*

$$\sum_{\substack{n \leq x \\ (n,a)=1}} n^s = \frac{\phi(a)x^{s+1}}{a(s+1)} + O(x^s \tau(a)).$$

Let f be a nonconstant polynomial with integral coefficients. If its decomposition into irreducible factors is $f = cg_1^{r_1} g_2^{r_2} \dots g_m^{r_m}$, then define $h(f) = \max_{1 \leq j \leq m} r_j$.

Lemma 6. *For every set F of nonconstant polynomials and for every $\varepsilon > 0$ we have*

$$N_F(n) = O(n^{u-h+\varepsilon}),$$

where $h = 1/h(f_1) + 1/h(f_2) + \dots + 1/h(f_u)$.

Proof. According to a result of R. Sitaramachandrarao and P. V. Krishnaiah [15, Lemma 3], $N_{f_i}(n) \leq C_i^{\omega(n)} n^{1-1/d_i}$ for every $n \in \mathbf{N}$, where d_i is the degree of f_i and $C_i > 0$ are constants. Observe that this result remains valid if we have $h(f_i)$ instead of d_i . Now using the familiar relation $C_i^{\omega(n)} = O(n^\varepsilon)$ for each $\varepsilon > 0$ and the definition of $N_F(n)$ we get the desired result.

Lemma 7. *For every set F of nonconstant polynomials, every subset S of \mathbf{N} , every regular convolution A and every $k, u \in \mathbf{N}$ we have*

$$\sum_{e \leq x} |\mu_{S,A}(e)| N_F(e^k) e^{-ku+\varepsilon/2} = \begin{cases} O(1), & \text{if } kh > 1 \text{ and } 0 < \varepsilon < kh - 1, \\ O(x^{1-kh+\varepsilon}), & \text{if } kh \leq 1 \text{ and } 0 < \varepsilon < kh, \end{cases}$$

$$\sum_{e > x} |\mu_{S,A}(e)| N_F(e^k) e^{-ku-1} = O(x^{\varepsilon-kh}), \quad \text{if } \varepsilon < kh.$$

Proof. By Lemmas 2 and 6 we have for every $\varepsilon > 0$

$$(9) \quad |\mu_{S,A}(e)| N_F(e^k) = O(e^{ku-kh+\varepsilon/2}),$$

$$\sum_{e \leq x} |\mu_{S,A}(e)| N_F(e^k) e^{-ku + \varepsilon/2} = O\left(\sum_{e \leq x} e^{-kh + \varepsilon}\right).$$

For $kh > 1$ and $\varepsilon < kh - 1$ we get $-kh + \varepsilon < -1$ and we use (4). For $kh \leq 1$ and $0 < \varepsilon < kh$ we have $-1 < -kh + \varepsilon < 0$ and we use (3).

By (9) we have with ε instead of $\varepsilon/2$

$$\sum_{e > x} |\mu_{S,A}(e)| N_F(e^k) e^{-ku - 1} = O\left(\sum_{e > x} e^{-kh - 1 + \varepsilon}\right),$$

where $-kh - 1 + \varepsilon < -1$ and applying (5) we get the second estimate.

Let A be a regular convolution such that $A = A_k$ for every $k \in \mathbf{N}$. Then for every prime p we have either $A(p^a) = \{1, p, p^2, \dots, p^a\} = D(p^a)$ or $A(p^a) = \{1, p^a\} = U(p^a)$ for every $a \in \mathbf{N}$, see [16, Theorem 3.3]. We say that A is a *cross-convolution*. Let P and Q be the set of the primes of the first and second kind of above, respectively. For $Q = \emptyset$ we have the Dirichlet convolution D and for $P = \emptyset$ we obtain the unitary convolution U .

Furthermore let $(P) = \{1\} \cup \{n \in \mathbf{N} : \text{each prime factor of } n \text{ belongs to } P\}$, $(Q) = \{1\} \cup \{n \in \mathbf{N} : \text{each prime factor of } n \text{ belongs to } Q\}$. It is clear that every $n \in \mathbf{N}$ can be written uniquely in the form $n = n_P n_Q$, where $n_P \in (P)$, $n_Q \in (Q)$ and $(n_P, n_Q) = 1$. In this case $A(n) = \{d : d|n, (d, n/d) \in (P)\}$ for every $n \in \mathbf{N}$.

Lemma 8. *If $s \geq 0$ and $a \in \mathbf{N}$, then*

$$\sum_{\substack{n \leq x \\ (n,a) \in (P)}} n^s = \frac{\phi(a_Q) x^{s+1}}{a_Q(s+1)} + O(V(x)),$$

where $V(x) = x^s$ or $x^s \tau(a)$, according as Q is finite or Q is infinite, respectively.

Proof. Observe that $(n, a) \in (P)$ if, and only if, $(n, \gamma(a_Q)) = 1$, where $\gamma(m)$ denotes the product of distinct prime factors of m . Hence

$$\sum_{\substack{n \leq x \\ (n,a) \in (P)}} n^s = \sum_{\substack{n \leq x \\ (n, \gamma(a_Q)) = 1}} n^s = \frac{\phi(\gamma(a_Q)) x^{s+1}}{\gamma(a_Q)(s+1)} + O(x^s \tau(\gamma(a_Q))),$$

by Lemma 5. Here $\phi(\gamma(a_Q))/\gamma(a_Q) = \phi(a_Q)/a_Q$ and if Q is finite, then $\tau(\gamma(a_Q)) \leq \tau(\prod_{p \in Q} p) = C$, a constant, which completes the proof.

Remark 3. We have $V(x) = O(x^s a^\varepsilon)$ for every Q and for every $\varepsilon > 0$.

Lemma 9. *For every set F of nonconstant polynomials, every regular A , every S and every $k, u \in \mathbf{N}$ the series*

$$\sum_{n=1}^{\infty} \frac{\mu_{S,A}(n) N_F(n^k) \phi(n_Q)}{n^{ku+1} n_Q}$$

is absolutely convergent. Let $\alpha_{F,S,A,k}$ denote its sum. If A is a cross-convolution and S is multiplicative, then

$$\alpha_{F,S,A,k} =$$

$$= \prod_{p \in P} \left(1 + \sum_{l=1}^{\infty} \frac{(\rho_S(p^l) - \rho_S(p^{l-1}))N_F(p^{lk})}{p^{l(ku+1)}}\right) \prod_{p \in Q} \left(1 + \left(1 - \frac{1}{p}\right) \sum_{l=1}^{\infty} \frac{(\rho_S(p^l) - 1)N_F(p^{lk})}{p^{l(ku+1)}}\right).$$

Proof. The absolute convergence of the series follows at once by Lemmas 2 and 6: the general term is

$$O(n^{ku-kh+\varepsilon}/n^{ku+1}) = O(1/n^{1+kh-\varepsilon}),$$

and we choose $\varepsilon < kh$.

If S is multiplicative, then the general term is multiplicative and the series can be expanded into an infinite product of Euler-type:

$$\begin{aligned} \alpha_{F,S,A,k} &= \prod_p \left(1 + \sum_{l=1}^{\infty} \frac{\mu_{S,A}(p^l)N_F(p^{lk})\phi((p^l)_Q)}{p^{l(ku+1)}(p^l)_Q}\right) \\ &= \prod_{p \in P} \left(1 + \sum_{l=1}^{\infty} \frac{\mu_{S,D}(p^l)N_F(p^{lk})}{p^{l(ku+1)}}\right) \prod_{p \in Q} \left(1 + \sum_{l=1}^{\infty} \frac{\mu_{S,U}(p^l)N_F(p^{lk})\phi(p^l)}{p^{l(ku+1)}p^l}\right) \\ &= \prod_{p \in P} \left(1 + \sum_{l=1}^{\infty} \frac{(\rho_S(p^l) - \rho_S(p^{l-1}))N_F(p^{lk})}{p^{l(ku+1)}}\right) \prod_{p \in Q} \left(1 + \sum_{l=1}^{\infty} \frac{(\rho_S(p^l) - 1)N_F(p^{lk})}{p^{l(ku+1)}}\left(1 - \frac{1}{p}\right)\right), \end{aligned}$$

by Remark 1.

Remark 4. If $S = \{1\}$ and A is a cross-convolution, then

$$\alpha_{F,S,A,k} = \prod_{p \in P} \left(1 - \frac{N_F(p^k)}{p^{ku+1}}\right) \prod_{p \in Q} \left(1 - \left(1 - \frac{1}{p}\right) \sum_{l=1}^{\infty} \frac{N_F(p^{lk})}{p^{l(ku+1)}}\right).$$

Theorem 4. *For every set F of nonconstant polynomials, every cross-convolution A , every S and every $k \in \mathbf{N}$ we have*

$$\sum_{n \leq x} \phi_{F,S,A,k}(n) = \frac{\alpha_{F,S,A,k}}{ku+1} x^{ku+1} + O(R(x)),$$

where $\alpha_{F,S,A,k}$ is defined in Lemma 9 and $R(x) = x^{ku}$ or $x^{ku-kh+\varepsilon+1}$, according as $kh > 1$ or $kh \leq 1$, respectively, for every $0 < \varepsilon < kh$.

Proof. Using Theorem 1 with $n_1 = n_2 = \dots = n_u = n$, Lemma 8 and Remark 3 we get for every $\varepsilon > 0$

$$\begin{aligned} \sum_{n \leq x} \phi_{F,S,A,k}(n) &= \sum_{n \leq x} \sum_{\substack{er=n \\ (e,r) \in (P)}} \mu_{S,A}(e)N_F(e^k)r^{ku} = \sum_{e \leq x} \mu_{S,A}(e)N_F(e^k) \sum_{\substack{r \leq x/e \\ (r,e) \in (P)}} r^{ku} \\ &= \sum_{e \leq x} \mu_{S,A}(e)N_F(e^k) \left(\frac{\phi(e_Q)(x/e)^{ku+1}}{e_Q(ku+1)} + O((x/e)^{ku}e^{\varepsilon/2})\right) \\ &= \frac{x^{ku+1}}{ku+1} \sum_{e \leq x} \frac{\mu_{S,A}(e)N_F(e^k)\phi(e_Q)}{e_Q e^{ku+1}} + O(x^{ku} \sum_{e \leq x} \frac{|\mu_{S,A}(e)|N_F(e^k)}{e^{ku-\varepsilon/2}}) \end{aligned}$$

$$= \frac{\alpha_{F,S,A,k}}{ku+1} x^{ku+1} + O(x^{ku+1} \sum_{e>x} \frac{|\mu_{S,A}(e)| N_F(e^k)}{e^{ku+1}}) + O(x^{ku} \sum_{e \leq x} \frac{|\mu_{S,A}(e)| N_F(e^k)}{e^{ku-\varepsilon/2}}).$$

Here the first O -term is $O(x^{ku+1} x^{\varepsilon-kh}) = O(x^{ku-kh+\varepsilon+1})$ for $\varepsilon < kh$ and the second O -term is $O(x^{ku})$ for $kh > 1$ and $\varepsilon < kh - 1$ and it is $O(x^{ku} x^{1-kh+\varepsilon}) = O(x^{ku-kh+\varepsilon+1})$ for $kh \leq 1$ and $\varepsilon < kh$, by Lemma 7, which proves the theorem.

Corollary 5. *For every set F of nonconstant polynomials, every cross-convolution A , every S and every k the average order of the function $\phi_{F,S,A,k}(n)$ is $\alpha_{F,S,A,k} n^{ku}$.*

For $n_1 = n_2 = \dots = n_u = n$ let $\phi_{S,A,k}(\mathbf{s}, \mathbf{d}, \mathbf{n}, n) = \phi_{S,A,k}(\mathbf{s}, \mathbf{d}, n)$.

Corollary 6. *If $(s_i, d_i^k)_k = 1$ for $i = 1, 2, \dots, u$ and A is a cross-convolution, then the average order of the function $\phi_{S,A,k}(\mathbf{s}, \mathbf{d}, n)$ is $\alpha_{S,A,k}(\mathbf{s}, \mathbf{d}) n^{ku}$, where*

$$\alpha_{S,A,k}(\mathbf{s}, \mathbf{d}) = \sum_{\substack{n=1 \\ (n,d)=1}}^{\infty} \frac{\mu_{S,A}(n) \phi(n_Q)}{n^{ku+1} n_Q},$$

$d = d_1 d_2 \dots d_u$. If in addition S is multiplicative, then

$$\alpha_{S,A,k}(\mathbf{s}, \mathbf{d}) = \prod_{\substack{p \in P \\ p \nmid d}} \left(1 + \sum_{l=1}^{\infty} \frac{\rho_S(p^l) - \rho_S(p^{l-1})}{p^{l(ku+1)}}\right) \prod_{\substack{p \in Q \\ p \mid d}} \left(1 + \left(1 - \frac{1}{p}\right) \sum_{l=1}^{\infty} \frac{\rho_S(p^l) - 1}{p^{l(ku+1)}}\right).$$

Proof. In this case $N_F(n^k) = \begin{cases} 1, & \text{if } (n, d_i) = 1 \text{ for } i = 1, 2, \dots, u \\ 0, & \text{otherwise} \end{cases}$ and we use Lemma 9.

Remark 5. If S is multiplicative and $A = D$, then

$$\alpha_{S,D,k}(\mathbf{s}, \mathbf{d}) = \frac{\zeta_S(ku+1)}{\zeta(ku+1) M_{S,d}(ku+1)},$$

where ζ is the Riemann zeta-function, $\zeta_S(z) = \sum_{n=1}^{\infty} \rho_S(n) n^{-z}$ (see [7]) and $M_{S,d}(z) = \prod_{p \mid d} (1 + \sum_{l=1}^{\infty} \mu_S(p^l) p^{-lz})$.

If $S = \{1\}$ and $A = D$, then

$$\alpha_{\{1\},D,k}(\mathbf{s}, \mathbf{d}) = \frac{d^{ku+1}}{\zeta(ku+1) \phi_{ku+1}(d)}.$$

If $S = \{1\}$ and $A = U$, then

$$\alpha_{\{1\},U,k}(\mathbf{s}, \mathbf{d}) = \prod_{p \mid d} \left(1 - \frac{p-1}{p(p^{ku+1}-1)}\right),$$

where the product is over all primes with $p \nmid d$.

Remark 6. The remainder term of the above asymptotic formula can be improved if we have more information on F , S and A , see [4], [10], [18], [20], [23].

As an example we prove

Theorem 5. If $(s_i, d_i^k)_k = 1$ and A is a cross-convolution, then

$$\sum_{n \leq x} \phi_{S,A,k}(\mathbf{s}, \mathbf{d}, n) = \frac{\alpha_{S,A,k}(\mathbf{s}, \mathbf{d})}{ku+1} x^{ku+1} + O(T(x)),$$

where $T(x) = x^{ku}$ ($ku > 1$), $x \log^2 x$ ($ku = 1$, Q is finite), $x \log^4 x$ ($ku = 1$, Q is infinite). If $ku = 1$ and S is multiplicative, then the error term can be improved into $T(x) = x \log x$ (Q is finite), $x \log^2 x$ (Q is infinite).

Proof. By Corollary 3 and Lemma 8 we get

$$\begin{aligned} \sum_{n \leq x} \phi_{S,A,k}(\mathbf{s}, \mathbf{d}, n) &= \sum_{\substack{e \leq x \\ (e,d)=1}} \mu_{S,A}(e) \sum_{\substack{r \leq x/e \\ (r,e) \in (P)}} r^{ku} \\ &= \sum_{\substack{e \leq x \\ (e,d)=1}} \mu_{S,A}(e) \left(\frac{\phi(e_Q)(x/e)^{ku+1}}{e_Q(ku+1)} + O(V(x/e)) \right) \\ &= \frac{x^{ku+1}}{ku+1} \sum_{\substack{e \leq x \\ (e,d)=1}} \frac{\mu_{S,A}(e) \phi(e_Q)}{e_Q e^{ku+1}} + O\left(\sum_{e \leq x} |\mu_{S,A}(e)| V(x/e)\right) \\ &= \frac{x^{ku+1}}{ku+1} \alpha_{S,A,k}(\mathbf{s}, \mathbf{d}) + O\left(x^{ku+1} \sum_{e > x} \frac{|\mu_{S,A}(e)|}{e^{ku+1}}\right) + O\left(\sum_{e \leq x} |\mu_{S,A}(e)| V(x/e)\right), \end{aligned}$$

where the first O -term is $O(x^{ku+1} \sum_{e > x} \tau(e) e^{-ku-1}) = O(x^{ku+1} x^{-ku} \log x) = O(x \log x)$, by Lemma 2 and (8). If S is multiplicative, then it is $O(x)$, by Lemma 1 and (5). If Q is finite, the second O -term is

$$\begin{aligned} &O\left(\sum_{e \leq x} |\mu_{S,A}(e)| (x/e)^{ku}\right) = O\left(x^{ku} \sum_{e \leq x} |\mu_{S,A}(e)| e^{-ku}\right) \\ &= \begin{cases} O(x^{ku} \sum_{e \leq x} \tau(e) e^{-ku}) = O(x^{ku}) & \text{for } ku > 1, \\ O(x \sum_{e \leq x} \frac{\tau(e)}{e}) = O(x \log^2 x) & \text{for } ku = 1, \\ O(x \sum_{e \leq x} \frac{1}{e}) = O(x \log x) & \text{for } ku = 1 \text{ and } S \text{ multiplicative,} \end{cases} \end{aligned}$$

by Lemmas 1 and 2, by (4) and (6).

If Q is infinite the second O -term is

$$\begin{aligned} &O\left(\sum_{e \leq x} |\mu_{S,A}(e)| (x/e)^{ku} \tau(e)\right) = O\left(x^{ku} \sum_{e \leq x} \frac{|\mu_{S,A}(e)| \tau(e)}{e^{ku}}\right) \\ &= \begin{cases} O(x^{ku} \sum_{e \leq x} \frac{\tau^2(e)}{e^{ku}}) = O(x^{ku}) & \text{for } ku > 1, \\ O(x \sum_{e \leq x} \frac{\tau^2(e)}{e}) = O(x \log^4 x) & \text{for } ku = 1, \\ O(x \sum_{e \leq x} \frac{\tau(e)}{e}) = O(x \log^2 x) & \text{for } ku = 1 \text{ and } S \text{ multiplicative,} \end{cases} \end{aligned}$$

by Lemmas 1 and 2, by (6) and (7).

Remark 7. For $k = 1, S = \{1\}, A = D$ and $k = 1, S = \{1\}, A = U$ this result was found by the first author [20, Theorem 2.4, Theorem 4.4].

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