The number and the sum of $P - k$-ary divisors of m which are prime to n

László Tóth
Department of Mathematics and Computer Science,
"Babeş-Bolyai" University,
Str. M. Kogălniceanu 1 , RO-3400 Cluj-Napoca, Romania,
E-mail: ltoth@math.ubbcluj.ro

Modern applied analysis (Ilieni, 1997)
Pure Math. Appl. (PU.M.A.) **9** *(1998), 221-232*

Abstract

Let P be an arbitrary set of positive primes and let k, m, d, n be positive integers. We say that d is a $P - k$-ary divisor of m if $d|m$ and each prime factor of $(d, m/d)_k$ belongs to P, where $(a, b)_k$ denotes the greatest common k-th power divisor of a and b. We establish asymptotic formulae for the sum of s-th powers (s real and $s \geq 1$) and for the number of $P - k$-ary divisors of m which are prime to n.

1 Introduction

Let $\mathbf{P}$ denote the set of the positive primes, let P be an arbitrary subset of $\mathbf{P}$ and Q be its complementary set. Denote by (P) and (Q) the multiplicative semigroups generated by $\{1\} \cup P$ and $\{1\} \cup Q$, respectively. Every $m \in \mathbf{N}, \mathbf{N}$ denoting the set of positive integers, can be written uniquely in the form $m = m_P m_Q$, where $m_P \in (P), m_Q \in (Q), (m_P, m_Q) = 1$. For $k, m, d \in \mathbf{N}$ we say that d is a $P - k$-ary divisor of m if $d|m$ and $(d, m/d)_k \in (P)$, i.e. each prime factor of $(d, m/d)_k$ belongs to P, where $(a, b)_k$ stands for the greatest common k-th power divisor of a and b.

Furthermore, if $n \in \mathbf{N}$ and $s \in \mathbf{R}, \mathbf{R}$ denoting the set of real numbers, let $\sigma_{P,k,s}(m, n)$ denote the sum of s-th powers of $P - k$-ary divisors of m which are prime to n. For $n = 1$, $\sigma_{P,k,s}(m, 1) \equiv \sigma_{P,k,s}(m)$ is the sum of s-th powers of all $P - k$-ary divisors of m. For $s = 0$, $\sigma_{P,k,0}(m, n) \equiv \tau_{P,k}(m, n)$ is the number of $P - k$-ary divisors of m which are prime to n and $\tau_{P,k}(m, 1) \equiv \tau_{P,k}(m)$ is the number of all $P - k$-ary divisors of m.

Observe that for $P = \mathbf{P}$ and $Q = \emptyset$, d is a $\mathbf{P} - k$-ary divisor of m if d is a usually divisor of m, $\sigma_{\mathbf{P},k,s}(m, n) \equiv \sigma_s(m, n)$ and $\sigma_{\mathbf{P},k,s}(m) \equiv \sigma_s(m)$ are the sum of s-th powers of divisors of m which are prime to n and the sum of s-th powers of all divisors of m, respectively, for every $k \in \mathbf{N}$. Let $\sigma_0(m, n) \equiv \tau(m, n)$ and $\tau(m, 1) \equiv \tau(m)$.

If $P = \emptyset$ and $Q = \mathbf{P}$, then d is an $\emptyset - k$-ary divisor of m if $d|m$ and $(d, m/d)_k = 1$, i.e. d is a k-ary divisor of m, and we obtain the functions $\sigma_{\emptyset,k,s}(m, n) \equiv \sigma^*_{k,s}(m, n)$ and $\sigma_{\emptyset,k,s}(m) \equiv \sigma^*_{k,s}(m)$ representing the sum of s-th powers of k-ary divisors of m which are prime to n and the sum of s-th powers of all k-ary divisors of m, respectively, investigated by J. Chidambaraswamy [2], D.Suryanarayana [13], D.Suryanarayana and V. Siva Rama Prasad [14].

The functions $\sigma^*_{k,0}(m,n) \equiv \tau^*_k(m,n)$ and $\tau^*_k(m,1) \equiv \tau^*_k(m)$ were studied by D. Suryanarayana [11], D.Suryanarayana and V. Siva Rama Prasad [15] and others. For $k=1$ we reobtain the notion of the unitary divisor of m and $\sigma^*_{1,s}(m) \equiv \sigma^*_s(m)$ is the sum of s-th powers of the unitary divisors of m, $\sigma^*_0(m) \equiv \tau^*(m)$ is the number of unitary divisors of m, cf. E. Cohen [3], [4].

Among many other special cases we mention only the following one. For $k=1$ and for an arbitrary $P \subseteq \mathbf{P}$ we obtain that d is a P-unitary divisor of m if $d|m$ and the gcd $(d, m/d) \in (P)$, i.e. d is an A-divisor of m, where $A = (P, Q)$ is a cross-convolution, introduced by us in [16], [17]. Note that the cross-convolution is a special case of Narkiewicz's regular convolution, cf. [7], [6]. Let $\sigma_{P,1,s}(m,n) \equiv \sigma_{A,s}(m,n), \sigma_{A,s}(m,1) \equiv \sigma_{A,s}(m)$ and $\sigma_{P,1,0}(m) \equiv \tau_A(m,n), \tau_A(m,1) \equiv \tau_A(m)$.

Arithmetical functions defined by cross-convolutions, including $\sigma_{A,s}(m)$ and $\tau_A(m)$, were investigated by us in [16], [17].

In this paper we establish asymptotic formulae for the summatory functions

$$\sum_{\substack{m\le x\\(m,u)=1}} \sigma_s(m,n), \qquad \sum_{\substack{m\le x\\(m,u)=1}} \sigma_{P,k,s}(m,n), \qquad \sum_{\substack{m\le x\\(m,u)=1}} \tau(m,n), \qquad \sum_{\substack{m\le x\\(m,u)=1}} \tau_{P,k}(m,n),$$

where $s \ge 1$ and $u \in \mathbf{N}$, which generalize and unify many known results, and obtain as particular cases asymptotic formulae for the functions discussed above. Our method is elementary and it applies an asymptotic estimate of B. Gordon and K. Rogers [5] regarding the divisor function $\tau(m)$.

2 Preliminaries

First we prove the following lemma.

Lemma 2.1 *For every $P \subseteq \mathbf{P}$ and for every $k, n \in \mathbf{N}, s \in \mathbf{R}$ the function $\sigma_{P,k,s}(m,n)$ is multiplicative in m.*

Proof. Let $K(m,d)$ be a function of two variables defined on the set $\{(m,d) \in \mathbf{N}\times\mathbf{N} : d|m\}$. The K-convolution of the arithmetical functions f and g is given by

$$(f *_K g)(m) = \sum_{d|m} K(m,d) f(d) g(m/d).$$

It is known, see [6], Chapter 4, that the K-convolution of two multiplicative arithmetical functions is multiplicative if and only if

$$K(m_1 m_2, d_1 d_2) = K(m_1, d_1) K(m_2, d_2)$$

for every $m_1, m_2, d_1, d_2 \in \mathbf{N}$ such that $(m_1, m_2) = 1, d_1|m_1, d_2|m_2$.

Now, let

$$K(m,d) = \begin{cases} 1, & \text{if } (d, m/d)_k \in (P), \\ 0, & \text{otherwise,} \end{cases}$$

$$I_n(m) = \begin{cases} 1, & \text{if } (m,n) = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $I_n(m)$ is (completely) multiplicative in m and we have

$$\sigma_{P,k,s}(m,n) = \sum_{d|m} K(m,d) I_n(d) d^s.$$

Therefore it is sufficient to prove that for every $m_1, m_2, d_1, d_2 \in \mathbf{N}$ with $(m_1, m_2) = 1$ and $d_1 | m_1, d_2 | m_2$ one has

$$\left(d_1 d_2, \frac{m_1 m_2}{d_1 d_2}\right)_k \in (P) \text{ if and only if } \left(d_1, \frac{m_1}{d_1}\right)_k \in (P) \text{ and } \left(d_2, \frac{m_2}{d_2}\right)_k \in (P). \tag{2.1}$$

Here

$$\left(\left(d_1, \frac{m_1}{d_1}\right), \left(d_2, \frac{m_2}{d_2}\right)\right) = 1,$$

hence

$$\left(d_1 d_2, \frac{m_1 m_2}{d_1 d_2}\right)_k = \left(d_1, \frac{m_1}{d_1}\right)_k \left(d_2, \frac{m_2}{d_2}\right)_k,$$

and this implies (2.1). □

Lemma 2.2 *For every $P \subseteq \mathbf{P}, k, n \in \mathbf{N}, s \in \mathbf{R}$ and for $m = \prod_{i=1}^r p_i^{a_i} > 1$ we have*

$$\sigma_{P,k,s}(m,n) = \prod_{\substack{p_i \in P \\ p_i \nmid n}} \frac{p_i^{s(a_i+1)} - 1}{p_i^s - 1} \prod_{\substack{p_i \in Q \\ p_i \nmid n \\ a_i < 2k}} \frac{p_i^{s(a_i+1)} - 1}{p_i^s - 1} \prod_{\substack{p_i \in Q \\ p_i \nmid n \\ a_i \geq 2k}} \frac{(p_i^{ks} - 1)(p_i^{s(a-k+1)} + 1)}{p_i^s - 1}, \tag{2.2}$$

where $s \neq 0$ and for $s = 0$,

$$\tau_{P,k}(m,n) = \prod_{\substack{p_i \in P \\ p_i \nmid n}} (a_i + 1) \prod_{\substack{p_i \in Q \\ p_i \nmid n \\ a_i < 2k}} (a_i + 1) \prod_{\substack{p_i \in Q \\ p_i \nmid n \\ a_i \geq 2k}} (2k). \tag{2.3}$$

Proof. By Lemma 2.1 it is sufficient to determine $\sigma_{P,k,s}(p^a, n)$ for every prime power p^a. If $p | n$, then $\sigma_{P,k,s}(p^a, n) = 1$ for each $a, n \in \mathbf{N}$. Now let $p \nmid n$. If $p \in P$, then $\sigma_{P,k,s}(p^a, n) = 1 + p^s + p^{2s} + ... + p^{as}$ which is $\frac{p^{s(a+1)}-1}{p^s-1}$ for $s \neq 0$ and it is $a + 1$ for $s = 0$. If $p \in Q$ and $a < 2k$, then for every $d | p^a$ we have $(d, p^a/d)_k = 1 \in (P)$, hence $\sigma_{P,k,s}(p^a, n) = 1 + p^s + p^{2s} + ... + p^{as}$. Finally, for $p \in Q$ and $a \geq 2k$ we have to consider the divisors p^b of p^a such that $(p^b, p^{a-b})_k = 1$, which holds for $b < k$ and for $b > a - k$, consequently $\sigma_{P,k,s}(p^a, n) = 1 + p^s + ... + p^{s(k-1)} + p^{s(a-k+1)} + ... + p^{as} = \frac{(p^{ks}-1)(p^{s(a-k+1)}+1)}{p^s-1}$ for $s \neq 0$ and it is $2k$ for $s = 0$. □

Remark. Observe that for each P and for each $k, m, n \in \mathbf{N}, s \in \mathbf{R}$,

$$\sigma_{P,k,s}(m,n) = \sigma(m_P, n)\sigma^*_{k,s}(m_Q, n),$$

$$\sigma_{P,k,s}(m) = \sigma_s(m_P)\sigma^*_{k,s}(m_Q).$$

Lemma 2.3 *For every $P \subseteq \mathbf{P}$ and for every $k, n, m \in \mathbf{N}, s \in \mathbf{R}$ we have*

$$\sigma_{P,k,s}(m,n) = \sum_{\substack{d^{2k}e = m \\ (d,n)=1}} h(d) d^{ks} \sigma_s(e, n), \tag{2.4}$$

where h is the multiplicative function defined by $h(1) = 1$ and

$$h(p^a) = \begin{cases} -1, & \text{if } p \in Q \text{ and } a = 1, \\ 0, & \text{otherwise}, \end{cases}$$

for every prime power p^a.

Proof. Both sides of (2.4) are multiplicative in m, cf. Lemma 2.1 and its proof, hence it is sufficient to prove it for $m = p^a$, a prime power. Let $F(m,n)$ denote the right hand side of (2.4). We have $F(p^a,n) = 1$ for every $p|n$. Furthermore, let $p \not| n$. Then for every $p \in P$, $F(p^a,n) = h(1)\sigma_s(p^a,n) = \sigma_s(p^a,n)$. Now, for $p \in Q$ we get: $F(p^a,n) = h(1)\sigma_s(p^a,n)+h(p)p^{ks}\sigma_s(p^{a-2k},n) = (1+p^s+...+p^{as})-p^{ks}(1+p^s+...+p^{s(a-2k)}) = 1+p^s+...+p^{s(k-1)}+p^{s(a-k+1)}+...+p^{as}$ if $a \geq 2k$ and $F(p^a,n) = \sigma_s(p^a,n)$ if $a < 2k$.

Therefore, $F(p^a,n) = \sigma_{P,k,s}(p^a,n)$, by Lemma 2.2, which completes the proof. □

3 Asymptotic formulae

In what follows all the constants implied by the O-symbols are independent of x, n and u.

Lemma 3.1 ([17], Lemma 5) *If the set Q is finite, then for $s > 0$ we have*

$$\sum_{\substack{n\leq x\\ n\in(Q)}} \frac{1}{n^s} = O(1), \tag{3.1}$$

$$\sum_{\substack{n> x\\ n\in(Q)}} \frac{1}{n^s} = O\left(\frac{1}{x^s}\right), \tag{3.2}$$

$$\sum_{\substack{n> x\\ n\in(Q)}} \frac{\log n}{n^s} = O\left(\frac{\log x}{x^s}\right). \tag{3.3}$$

Remark. Compare the above estimates with the following well-known formulae:

$$\sum_{n\leq x} \frac{1}{n^s} = \begin{cases} O(\log x), & \text{if } s=1,\\ O(1), & \text{if } s>1,\end{cases} \tag{3.4}$$

$$\sum_{n> x} \frac{1}{n^s} = O\left(\frac{1}{x^{s-1}}\right), \quad s>1, \tag{3.5}$$

$$\sum_{n> x} \frac{\log n}{n^s} = O\left(\frac{\log x}{x^{s-1}}\right), \quad s>1. \tag{3.6}$$

We need the following two lemmas which are generalizations of certain results of D. Suryanarayana [12], see also [17].

Lemma 3.2 *If h is the function defined in Lemma 2.3, $u \in \mathbf{N}$ and $s > 1$, then*

$$\sum_{\substack{n\leq x\\ (n,u)=1}} \frac{h(n)}{n^s} = \frac{u_Q^s}{\zeta_Q(s)\phi_s(u_Q)} + O(A_s(x,Q)), \tag{3.7}$$

where

$$\zeta_Q(s) = \sum_{\substack{n=1\\ n\in(Q)}}^{\infty} \frac{1}{n^s}, \qquad \phi_s(n) = \sum_{d|n} d^s\mu(n/d),$$

μ denoting the Möbius function and $A_s(x,Q) = x^{-s}$ (Q finite) x^{1-s} (Q infinite).

Proof. We have

$$\sum_{\substack{n\le x\\(n,u)=1}} \frac{h(n)}{n^s} = \sum_{\substack{n=1\\(n,u)=1}}^{\infty} \frac{h(n)}{n^s} + O\left(\sum_{n>x} \frac{|h(n)|}{n^s}\right).$$

Here the series is absolutely convergent and its sum is $\frac{u_Q^s}{\zeta_Q(s)\phi_s(u_Q)}$ using the Euler product formula. The O-term is

$$O\left(\sum_{\substack{n>x\\n\in(Q)}} \frac{1}{n^s}\right) = O(A_s(x,Q))$$

by (3.2) and (3.5). □

Lemma 3.3 *If h is the function defined in Lemma 2.3, $u\in\mathbf{N}$ and $s>1$, then*

$$\sum_{\substack{n\le x\\(n,u)=1}} \frac{h(n)\log n}{n^s} = \frac{u_Q^s}{\zeta_Q(s)\phi_s(u_Q)}\left(\alpha_s(u_Q)+\frac{\zeta_Q'(s)}{\zeta_Q(s)}\right) + O(A_s(x,Q)\log x), \tag{3.8}$$

where ζ_Q' is the derivative of ζ_Q, $A_s(x,Q)$ is given in Lemma 3.2 and

$$\alpha_s(u) = \sum_{p|u} \frac{\log p}{p^s-1}.$$

Proof. We have

$$\sum_{\substack{n\le x\\(n,u)=1}} \frac{h(n)\log n}{n^s} = \sum_{\substack{n=1\\(n,u)=1}}^{\infty} \frac{h(n)\log n}{n^s} + O\left(\sum_{n>x} \frac{|h(n)|\log n}{n^s}\right),$$

where the series is uniformly convergent for $s\ge 1+\varepsilon>1$ and its sum can be obtained by termwise differentiation with respect to s of the series given in the proof of Lemma 3.2, see [12], Lemma 2.5 for $h=\mu$. The O-term is

$$O\left(\sum_{\substack{n>x\\n\in(Q)}} \frac{\log n}{n^s}\right) = O(A_s(x,Q)\log x)$$

by (3.3) and (3.6). □

We also need the following results.

Lemma 3.4 ([5], Lemma 3, see also [10], Lemma 2) *If $u\in\mathbf{N}$, then*

$$\sum_{\substack{n\le x\\(n,u)=1}} \tau(n) = \left(\frac{\phi(u)}{u}\right)^2 x(\log x + 2C - 1 + 2\alpha(u)) + O(S(u)\sqrt{x}), \tag{3.9}$$

where $\phi\equiv\phi_1$ is the Euler totient function, C is Euler's constant and

$$\alpha(u)\equiv\alpha_1(u) = \sum_{p|u}\frac{\log p}{p-1}, \qquad S(u) = \sum_{d|u}\frac{3^{\omega(d)}}{\sqrt{d}}.$$

Lemma 3.5 *If $s \geq 1$ and $u \in \mathbf{N}$, then*

$$\sum_{\substack{m\leq x\\(m,u)=1}} \sigma_s(m) = \frac{\zeta(s+1)\phi(u)\phi_{s+1}(u)}{(s+1)u^{s+2}}x^{s+1} + O\left(\frac{\tau^*(u)u^s}{\phi_s(u)}R_s(x)\right), \tag{3.10}$$

where ζ is the Riemann zeta function and $R_s(x) = x^s$ $(s > 1)$, $x(log x)^{2/3}$ $(s = 1)$.

Proof. For $s = 1$ this result is exactly Lemma 2.2 of [8] and the same proof works out for $s > 1$, see also [9]. $\square$

Now we are ready to prove the following asymptotic formulae.

Theorem 3.1 *If $n, u \in \mathbf{N}$, then*

$$\sum_{\substack{m\leq x\\(m,u)=1}} \tau(m,n) = \left(\frac{\phi(u)}{u}\right)^2 f(n,u)x(\log x + 2C - 1 + 2\alpha(u) + \alpha(n,u))$$

$$+O(\sigma^*_{-1/2}(n,u)S(u)\sqrt{x}), \tag{3.11}$$

where

$$f(n,u) = \prod_{\substack{p|n\\p\nmid u}}\left(1-\frac{1}{p}\right), \qquad \alpha(n,u) = \sum_{\substack{p|n\\p\nmid u}}\frac{\log p}{p-1},$$

$\sigma^*_r(n,u)$ is the sum of r-th powers ot the unitary divisors of n which are prime to u, and for $s \geq 1$,

$$\sum_{\substack{m\leq x\\(m,u)=1}} \sigma_s(m,n) = \frac{\zeta(s+1)\phi(u)\phi_{s+1}(u)f(n,u)}{(s+1)u^{s+2}}x^{s+1} + O\left(\frac{\tau^*(u)u^s}{\phi_s(u)}\tau^*(n,u)R_s(x)\right), \tag{3.12}$$

where $\tau^*(n,u) \equiv \sigma^*_0(n,u)$ and $R_s(x)$ is defined in Lemma 3.5.

Proof. We have

$$\sum_{\substack{m\leq x\\(m,u)=1}} \sigma_s(m,n) = \sum_{\substack{m\leq x\\(m,u)=1}}\sum_{\substack{de=m\\(d,n)=1}} d^s = \sum_{\substack{de\leq x\\(de,u)=1}} d^s \sum_{r|(d,n)}\mu(r)$$

$$= \sum_{\substack{r|n\\(r,u)=1}}\mu(r)r^s \sum_{\substack{qe\leq x/r\\(qe,u)=1}} q^s = \sum_{\substack{r|n\\(r,u)=1}}\mu(r)r^s\sum_{\substack{t\leq x/r\\(t,u)=1}}\sigma_s(t).$$

For $s = 0$ we get using Lemma 3.4,

$$\sum_{\substack{m\leq x\\(m,u)=1}} \tau(m,n) = \sum_{\substack{r|n\\(r,u)=1}}\mu(r)\left(\left(\frac{\phi(u)}{u}\right)^2\frac{x}{r}\left(\log\frac{x}{r} + 2C - 1 + 2\alpha(u)\right) + O\left(\sqrt{\frac{x}{r}}S(u)\right)\right)$$

$$= \left(\frac{\phi(u)}{u}\right)^2 x\left(\left(\sum_{\substack{r|n\\(r,u)=1}}\frac{\mu(r)}{r}\right)(\log x + 2C - 1 + 2\alpha(u)) - \sum_{\substack{r|n\\(r,u)=1}}\frac{\mu(r)\log r}{r}\right.$$

$$+O\left(\sqrt{x}S(u)\sum_{\substack{r|n\\(r,u)=1}}\frac{\mu^2(r)}{\sqrt{r}}\right)$$

$$=\left(\frac{\phi(u)}{u}\right)^2 f(n,u)x(\log x+2C-1+2\alpha(u)+\alpha(n,u))+O(\sigma^*_{-1/2}(n,u)S(u)\sqrt{x}),$$

where

$$-\sum_{\substack{r|n\\(r,u)=1}}\frac{\mu(r)\log r}{r}=f(n,u)\alpha(n,u). \qquad (3.13)$$

Here, in order to show (3.13) we use that

$$-\sum_{r|n}\frac{\mu(r)\log r}{r}=\frac{\phi(n)}{n}\sum_{p|n}\frac{\log p}{p-1}, \quad n\in\mathbf{N},$$

cf. [1]. Denoting by $n(u)$ the greatest divisor of n which is prime to u we have

$$-\sum_{\substack{r|n\\(r,u)=1}}\frac{\mu(r)\log r}{r}=-\sum_{r|n(u)}\frac{\mu(r)\log r}{r}=\frac{\phi(n(u))}{n(u)}\sum_{p|n(u)}\frac{\log p}{p-1}$$

$$=\prod_{p|n(u)}(1-\frac{1}{p})\sum_{p|n(u)}\frac{\log p}{p-1}=f(n,u)\alpha(n,u),$$

which proves (3.11).

Now, for $s\geq 1$ we use Lemma 3.5:

$$\sum_{\substack{m\leq x\\(m,u)=1}}\sigma_s(m,n)=\sum_{\substack{r|n\\(r,u)=1}}\mu(r)r^s\left(\frac{\zeta(s+1)\phi(u)\phi_{s+1}(u)}{(s+1)u^{s+2}}\left(\frac{x}{r}\right)^{s+1}+O\left(R_s(\frac{x}{r})\frac{\tau^*(u)u^s}{\phi_s(u)}\right)\right)$$

$$=\frac{\zeta(s+1)\phi(u)\phi_{s+1}(u)}{(s+1)u^{s+2}}x^{s+1}\sum_{\substack{r|n\\(r,u)=1}}\frac{\mu(r)}{r}+O\left(\frac{\tau^*(u)u^s}{\phi_s(u)}\sum_{\substack{r|n\\(r,u)=1}}\mu^2(r)r^sR_s(\frac{x}{r})\right).$$

Using the definition of $R_s(x)$ we get at once the O term given in (3.12), which completes the proof. □

Remark. Note that for $u=1$, $f(n,1)=\phi(n)/n, \alpha(n,1)=\alpha(n)$ for each $n\in\mathbf{N}$ and for $n=1$, $f(1,u)=1, \alpha(1,u)=0$ for each $u\in\mathbf{N}$.

For $u=1$ formula (3.11) is due to D. Suryanarayana and V. Siva Rama Prasad [15], theorem 4.1.

Theorem 3.2 *If $P\subseteq\mathbf{P}$ and $k,n,u\in\mathbf{N}$, then*

$$\sum_{\substack{m\leq x\\(m,u)=1}}\tau_{P,k}(m,n)=\left(\frac{\phi(u)}{u}\right)^2 f(n,u)\frac{(nu)_Q^{2k}}{\zeta_Q(2k)\phi_{2k}((nu)_Q)}x\,(\log x+2C-1+2\alpha(u)+\alpha(n,u)$$

$$\left.-2k\alpha_{2k}((nu)_Q)-2k\frac{\zeta'_Q(2k)}{\zeta_Q(2k)}\right)+O(\sigma^*_{-1/2}(n,u)S(u)H_k(x,Q)), \qquad (3.14)$$

where $H_k(x,Q)=\sqrt{x}$ *($k\ge 2$ or $k=1$ and Q finite),* $\sqrt{x}\log x$ *($k=1$ and Q infinite), and for* $s\ge 1$,

$$\sum_{\substack{m\le x\\(m,u)=1}}\sigma_{P,k,s}(m,n)=\frac{\zeta(s+1)\phi(u)\phi_{s+1}(u)f(n,u)(nu)_Q^{k(s+2)}}{(s+1)\zeta_Q(k(s+2))u^{s+2}\phi_{k(s+2)}((nu)_Q)}x^{s+1}$$

$$+O\left(\frac{\tau^*(u)u^s}{\phi_s(u)}\tau^*(n,u)R_{k,s}(x,Q)\right), \tag{3.15}$$

where $R_{k,s}(x,Q)=x^s$ *($s>1$)* $x(\log x)^{5/3}$ *($s=1,k=1$ and Q infinite)* $x(\log x)^{2/3}$ *($s=1$ and $k\ge 2$ or $s=1,k=1$ and Q finite).*

Proof. Using Lemma 2.3 we obtain

$$\sum_{\substack{m\le x\\(m,u)=1}}\sigma_{P,k,s}(m,n)=\sum_{\substack{d^{2k}e\le x\\(d,n)=1\\(d,u)=(e,u)=1}}h(d)d^{ks}\sigma_s(e,n)=\sum_{\substack{d\le\sqrt[2k]{x}\\(d,nu)=1}}h(d)d^{ks}\sum_{\substack{e\le x/d^{2k}\\(e,u)=1}}\sigma_s(e,n).$$

For $s=0$ we have by (3.11)

$$\sum_{\substack{m\le x\\(m,u)=1}}\tau_{P,k}(m,n)$$

$$=\sum_{\substack{d\le\sqrt[2k]{x}\\(d,nu)=1}}h(d)\left(\left(\frac{\phi(u)}{u}\right)^2 f(n,u)\frac{x}{d^{2k}}\left(\log\frac{x}{d^{2k}}+2C-1+2\alpha(u)+\alpha(n,u)\right)\right.$$

$$\left.+O\left(\sqrt{\frac{x}{d^{2k}}}S(u)\sigma^*_{-1/2}(n,u)\right)\right)$$

$$=\left(\frac{\phi(u)}{u}\right)^2 f(n,u)x\left((\log x+2C-1+2\alpha(u)+\alpha(n,u))\sum_{\substack{d\le\sqrt[2k]{x}\\(d,nu)=1}}\frac{h(d)}{d^{2k}}\right.$$

$$\left.-2k\sum_{\substack{d\le\sqrt[2k]{x}\\(d,nu)=1}}\frac{h(d)\log d}{d^{2k}}\right)+O\left(S(u)\sigma^*_{-1/2}(n,u)\sqrt{x}\sum_{d\le\sqrt[2k]{x}}\frac{|h(d)|}{d^k}\right).$$

Now (3.14) yields by Lemmas 3.2 and 3.3 applied for $s=2k>1$ and by (3.1) and (3.4).

Furthermore, in case $s\ge 1$ we get by (3.12)

$$\sum_{\substack{m\le x\\(m,u)=1}}\sigma_{P,k,s}(m,n)=\sum_{\substack{d\le\sqrt[2k]{x}\\(d,nu)=1}}h(d)d^{ks}\left(\frac{\zeta(s+1)\phi(u)\phi_{s+1}(u)f(n,u)}{(s+1)u^{s+2}}\left(\frac{x}{d^{2k}}\right)^{s+1}\right.$$

$$\left.+O\left(R_s\left(\frac{x}{d^{2k}}\right)\frac{\tau^*(u)u^s}{\phi_s(u)}\tau^*(n,u)\right)\right)$$

$$=\frac{\zeta(s+1)\phi(u)\phi_{s+1}(u)f(n,u)}{(s+1)u^{s+2}}x^{s+1}\sum_{\substack{d\le\sqrt[2k]{x}\\(d,nu)=1}}\frac{h(d)}{d^{k(s+2)}}$$

$$+O\left(\frac{\tau^*(u)u^s}{\phi_s(u)}\tau^*(n,u)\sum_{\substack{d\le \sqrt[2k]{x}\\ (d,nu)=1}}|h(d)|d^{ks}R_s\left(\frac{x}{d^{2k}}\right)\right).$$

Here we apply Lemma 3.2, (3.1), (3.4) and get the desired formula (3.15). □

Remark In case $P=\emptyset, Q=\mathbf{P}$, h is the Möbius μ function and the error term of (3.14) can be improved using certain estimates regarding the μ function with and without assuming the Riemann hypothesis, see D. Suryanarayana and V. Siva Rama Prasad [15].

Corollary 3.1 $(u=1)$ *If* $P\subseteq \mathbf{P}$ *and* $k,n\in\mathbf{N}$, *then*

$$\sum_{m\le x}\tau_{P,k}(m,n)=\frac{n_Q^{2k}\phi(n)}{\zeta_Q(2k)n\phi_{2k}(n_Q)}x\left(\log x+2C-1+\alpha(n)\right.$$

$$\left.-2k\alpha_{2k}(n_Q)-2k\frac{\zeta_Q'(2k)}{\zeta_Q(2k)}\right)+O(\sigma^*_{-1/2}(n)H_k(x,Q)), \tag{3.16}$$

and for $s\ge 1$,

$$\sum_{m\le x}\sigma_{P,k,s}(m,n)=\frac{\zeta(s+1)\phi(n)n_Q^{k(s+2)}}{(s+1)n\zeta_Q(k(s+2))\phi_{k(s+2)}(n_Q)}x^{s+1}$$

$$+O\left(\tau^*(n)R_{k,s}(x,Q)\right). \tag{3.17}$$

For $P=\emptyset, Q=\mathbf{P}$ formula (3.16) is due to D. Suryanarayana and V. Siva Rama Prasad [15], Theorem 4.3 with better O-term.

Corollary 3.2 $(n=1)$ *If* $P\subseteq \mathbf{P}$ *and* $k,u\in\mathbf{N}$, *then*

$$\sum_{\substack{m\le x\\ (m,u)=1}}\tau_{P,k}(m)=\left(\frac{\phi(u)}{u}\right)^2\frac{u_Q^{2k}}{\zeta_Q(2k)\phi_{2k}(u_Q)}x\left(\log x+2C-1+2\alpha(u)\right.$$

$$\left.-2k\alpha_{2k}(u_Q)-2k\frac{\zeta_Q'(2k)}{\zeta_Q(2k)}\right)+O(S(u)H_k(x,Q)), \tag{3.18}$$

and for $s\ge 1$,

$$\sum_{\substack{m\le x\\ (m,u)=1}}\sigma_{P,k,s}(m)=\frac{\zeta(s+1)\phi(u)\phi_{s+1}(u)u_Q^{k(s+2)}}{(s+1)\zeta_Q(k(s+2))u^{s+2}\phi_{k(s+2)}(u_Q)}x^{s+1}$$

$$+O\left(\frac{\tau^*(u)u^s}{\phi_s(u)}R_{k,s}(x,Q)\right). \tag{3.19}$$

For $P=\emptyset, Q=\mathbf{P}$ and $u=1$ formula (3.19) is due to J. Chidambaraswamy [2], Corollary 3.1.3 and to D. Suryanarayana and V. Siva Rama Prasad [14], Corollary 4.1.1, see also D. Suryanarayana [13], Corollary 3.1 (case $s=1$).

Corollary 3.3 $(k=1)$ *If* $P \subseteq \mathbf{P}$ *and* $n, u \in \mathbf{N}$, *then*

$$\sum_{\substack{m \le x \\ (m,u)=1}} \tau_A(m,n) = \left(\frac{\phi(u)}{u}\right)^2 f(n,u) \frac{(nu)_Q^2}{\zeta_Q(2)\phi_2((nu)_Q)} x \left(\log x + 2C - 1 + 2\alpha(u) + \alpha(n,u)\right.$$

$$\left. -2\alpha_2((nu)_Q) - 2\frac{\zeta_Q'(2)}{\zeta_Q(2)}\right) + O(\sigma^*_{-1/2}(n,u)S(u)H(x,Q)), \tag{3.20}$$

where $H(x,Q) = \sqrt{x}$ *(Q finite),* $\sqrt{x}\log x$ *(Q infinite), and for* $s \ge 1$,

$$\sum_{\substack{m \le x \\ (m,u)=1}} \sigma_{A,s}(m,n) = \frac{\zeta(s+1)\phi(u)\phi_{s+1}(u) f(n,u)(nu)_Q^{s+2}}{(s+1)\zeta_Q(s+2)u^{s+2}\phi_{s+2}((nu)_Q)} x^{s+1}$$

$$+O\left(\frac{\tau^*(u)u^s}{\phi_s(u)}\tau^*(n,u)R_s(x,Q)\right), \tag{3.21}$$

where $R_s(x,Q) = x^s$ $(s>1)$ $x(\log x)^{5/3}$ $(s=1$ *and* Q *infinite)* $x(\log x)^{2/3}$ $(s=1$ *and* Q *finite).*

For $n=1$ formula (3.20) was given by us in [17], Theorem 1. For $n=u=1$ (3.21) appears in [16], Theorems 7 and 12.

For $A=U, n=s=1$ formula (3.21) is due to V. Sitaramaiah and M. V. Subbarao [8], Lemma 2.4.

References

[1] G. M. Bergmann, *Solution of problem 5091*, Amer. Math. Monthly 71 (1964), 334-335.

[2] J. Chidambaraswamy, *The K-unitary convolution of certain arithmetical functions*, Publ. Math. Debrecen 17 (1970), 67-74.

[3] E. Cohen, *Arithmetical functions associated with the unitary divisors of an integer*, Math. Z. 74 (1960), 66-80.

[4] E. Cohen, *The number of unitary divisors of an integer*, Amer. Math. Monthly 67 (1960), 879-880.

[5] B. Gordon, K. Rogers, *Sums of the divisor function*, Canadian J. Math. 16 (1964), 151-158.

[6] P. J. McCarthy, *Introduction to arithmetical functions*, Springer- Verlag, New York, Berlin, Heidelberg, Tokyo, 1986.

[7] W. Narkiewicz, *On a class of arithmetical convolutions*, Colloq. Math. 10 (1963), 81-94.

[8] V. Sitaramaiah, M. V. Subbarao, *Asymptotic formulae for sums of reciprocals of some multiplicative functions*, J. Indian Math. Soc. 57 (1991), 153-167.

[9] V. Sita Ramaiah, D. Suryanarayana, *An order result involving the* σ*-function*, Indian J. Pure Appl. Math. 12 (1981), 1192-1200.

[10] M. V. Subbarao, D. Suryanarayana, *Sums of the divisor and unitary divisor functions*, J. Reine Angew. Math. 302 (1978), 1-15.

[11] D. Suryanarayana, *The number of k-ary divisors of an integer*, Monatsh. Math. 72 (1968), 445-450.

[12] D. Suryanarayana, *The greatest divisor of n which is prime to k*, Math. Student 37 (1969), 147-157.

[13] D. Suryanarayana, *Some theorems concerning the k-ary divisors of an integer*, Math. Student 39 (1971), 384-394.

[14] D. Suryanarayana, V. Siva Rama Prasad, *Sum functions of k-ary and semi-k-ary divisors*, J. Austr. Math. Soc. 15 (1973), 148-162.

[15] D. Suryanarayana, V. Siva Rama Prasad *The number of k-free and k-ary divisors of m which are prime to n*, J. Reine Angew. Math. 264 (1973), 56-75.

[16] L. Tóth, *Asymptotic formulae concerning arithmetical functions defined by cross-convolutions, I. Divisor-sum functions and Euler-type functions*, Publ. Math. Debrecen 50 (1997), 159-176.

[17] L. Tóth, *Asymptotic formulae concerning arithmetical functions defined by cross-convolutions, II. The divisor function*, Studia Univ. Babeş-Bolyai, Math., to appear.

AMS Subject Classification: 11A25, 11N37

Key Words and Phrases: divisor function, k-ary divisor, Narkiewicz's regular convolution, asymptotic formula