On the asymptotic densities of certain subsets of $\mathbf{N}^k$

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Abstract

We determine the asymptotic density δ_k of the set of ordered k-tuples $(n_1, ..., n_k) \in \mathbf{N}^k, k \geq 2$, such that there exists no prime power p^a, $a \geq 1$, appearing in the canonical factorization of each n_i, $1 \leq i \leq k$, and deduce asymptotic formulae with error terms regarding this problem and analogous ones. We give numerical approximations of the constants δ_k and improve the error term of formula (1.2) due to E. Cohen.

We point out that our treatment, based on certain inversion functions, is applicable also in case $k = 1$ in order to establish asymptotic formulae with error terms regarding the densities of subsets of $\mathbf{N}$ with additional multiplicative properties. These generalize an often cited result of G. J. Rieger.

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1 Introduction

Let $k \geq 2$ be a fixed integer. What is the asymptotic density δ_k of the set of ordered k-tuples $(n_1, ..., n_k) \in \mathbf{N}^k$, such that there exists no prime power p^a, $a \geq 1$, appearing in the canonical factorization of each n_i, $1 \leq i \leq k$?

This problem is analogous to the following one: What is the asymptotic density d_k of the set of k-tuples which are relatively prime, i.e. k-tuples $(n_1, ..., n_k) \in \mathbf{N}^k$ such that there exists no prime p, appearing in the canonical factorization of each n_i, $1 \leq i \leq k$?

It is known that $d_k = 1/\zeta(k)$, where ζ is the Riemann zeta function, and this value can be considered as the probability that k integers ($k \geq 2$) chosen at random are relatively prime. More precisely,

$$(1.1) \qquad N_k(x) := \#\{(n_1, ..., n_k) \in (\mathbf{N} \cap [1, x])^k : gcd(n_1, ..., n_k) = 1\} = \frac{1}{\zeta(k)} x^k + R_k(x),$$

where $R_k(x) = O(x^{k-1})$ for $k > 2$, $R_2(x) = O(x \log x)$ for $k = 2$, and $d_k = \lim_{x\to\infty} N_k(x)/x^k = 1/\zeta(k)$. This result goes back to the work of J. J. Sylvester [9] and D. N. Lehmer [3], see also J. E. Nymann [5].

There are several generalizations of (1.1) in the literature. For example, let S be an arbitrary subset of N. Then

$$(1.2)\quad N_k(x,S) := \#\{(n_1,...,n_k) \in (\mathbf{N}\cap[1,x])^k : gcd(n_1,...,n_k) \in S\} = \frac{\zeta_S(k)}{\zeta(k)}x^k + T_k(x),$$

where

$$\zeta_S(k) = \sum_{\substack{n=1\\ n\in S}}^{\infty} \frac{1}{n^k}$$

and $T_k(x) = O(x^{k-1})$ for $k > 2$, $T_2(x) = O(x\log^2 x)$ for $k = 2$, for every $S \subseteq \mathbf{N}$, due to E. Cohen [1]. Therefore the asymptotic density of the set of ordered k-tuples $(n_1,...,n_k)$ for which $gcd(n_1,...,n_k)$ belongs to S is $\lim_{x\to\infty} N_k(x,S)/x^k = \frac{\zeta_S(k)}{\zeta(k)}$.

J. E. Nymann [6] shows that if the characteristic function ρ_S of $\emptyset \neq S \subseteq \mathbf{N}$ is completely multiplicative and if $\#\{n : n \in S\cap[1,x]\} = Ax + O(1)$, where A is the asymptotic density of S, then

$$(1.3)\quad \#\{(n_1,...,n_k) \in (S\cap[1,x])^k : gcd(n_1,...,n_k) = 1\} = A^k \prod_{p\in S}\left(1-\frac{1}{p^k}\right)x^k + R_k(x),$$

where $R_k(x)$ is the same as above. Therefore, if $P_k^S(n)$ denotes the probability that k integers $(k \geq 2)$ chosen at random from $S\cap[1,n]$ are relatively prime, then

$$\lim_{n\to\infty} P_k^S(n) = \prod_{p\in S}\left(1-\frac{1}{p^k}\right).$$

This result can be applied for $S = \{n : gcd(n, p_1\cdots p_r) = 1\}$, where $\{p_1,...,p_r\}$ is a given finite set of distinct primes.

Now return to the problem at the beginning. It is obvious that $\delta_k \geq d_k = 1/\zeta(k)$ for every $k \geq 2$ and thus $\lim_{k\to\infty}\delta_k = 1$. Which is the exact value of δ_k ?

In order to solve this problem we use the concept of the unitary divisor. For $d,n \in \mathbf{N}$, d is called a unitary divisor (or block divisor) of n if $d|n$ and $gcd(d,n/d) = 1$, notation $d||n$. Various other problems concerning unitary divisors, including properties of arithmetical functions and arithmetical convolutions defined by unitary divisors, have been studied extensively in the literature, see for example [4] and its bibliography. Denote the greatest common unitary divisor of $n_1,...,n_k$ by $gcud(n_1,...,n_k)$.

Our question can be reformulated in this way: What is the asymptotic density δ_k of the set of ordered k-tuples $(n_1,...,n_k)$ such that $gcud(n_1,...,n_k) = 1$, or more generally, $gcud(n_1,...,n_k) \in S$?

Furthermore, what is the probability that for k integers $n_1,...,n_k$ chosen at random from $S\cap[1,n]$ one has $gcud(n_1,...,n_k) = 1$?

In this paper we determine the value δ_k and deduce asymptotic formulae with error termes analogous to (1.1) -(1.3), regarding these problems. We give numerical approximations of the constants δ_k and also improve the error term of (1.2) of E. Cohen.

The treatment we use is based on on the inversion functions μ_S^* and μ_S attached to the subset S. We point out that this is applicable also in case $k = 1$ in order to establish asymptotics regarding the densities of certain subsets S of $\mathbf{N}$, generalizing in this way an often cited result of G. J. Rieger [7].

Note that the value δ_2 is given by D. SURYANARAYANA and M. V. SUBBARAO [8], Corollary 3.6.3, applying other arguments as those of the present paper.

Using the concept of regular cross-convolution, see [11], [12], it is possible to deduce more general results, including (1.1) - (1.3) and (2.1) and (2.4) of this paper. We do not go into details.

2 Results

Let $S \subseteq \mathbf{N}$. We say that S is (completely) multiplicative if $1 \in S$ and its characteristic function $\rho_S(n)$ is (completely) multiplicative. Define the function $\mu_S^*(n)$ by

$$\sum_{d||n} \mu_S^*(d) = \rho_S(n), \quad n \in \mathbf{N},$$

that is

$$\mu_S^*(n) = \sum_{d||n} \rho_S(d)\mu^*(n/d), \quad n \in \mathbf{N},$$

where the sums are extended over the unitary divisors of n and $\mu^*(n) := \mu_{\{1\}}^*(n) = (-1)^{\omega(n)}$, $\omega(n)$ denoting the number of distinct prime factors of n.

Furthermore, let $\phi(n)$ and $\theta(n)$ denote Euler's function and the number of squarefree divisors of n, respectively.

Theorem 2.1 *If $k \geq 2$ and S is an arbitrary subset of $\mathbf{N}$, then*

$$\#\{(n_1, ..., n_k) \in (\mathbf{N} \cap [1, x])^k : gcud(n_1, ..., n_k) \in S\} = \delta_k(S)x^k + V_k(x, S), \tag{2.1}$$

where

$$\delta_k(S) = \sum_{n=1}^{\infty} \frac{\mu_S^*(n)\phi^k(n)}{n^{2k}}$$

and the remainder term can be evaluated as follows:

(1) $V_k(x, S) = O(x^{k-1})$ for $k > 2$ and for an arbitrary S,

(2) $V_2(x, S) = O(x \log^4 x)$ for an arbitrary S,

(3) $V_2(x, S) = O(x \log^2 x)$ for an S such that $\sum_{n\in S} \frac{\theta(n)}{n} < \infty$ (in particular for every finite S) and for every multiplicative S,

(4) $V_2(x, S) = O(x)$ for every multiplicative S such that $\sum_{p\notin S} \frac{1}{p} < \infty$ (in particular if the set $\{p : p \notin S\}$ is finite).

If S is multiplicative, then

$$\delta_k(S) = \prod_p \left(1 - (1 - \frac{1}{p})^k \sum_{\substack{a=1 \\ p^a \notin S}}^{\infty} \frac{1}{p^{ak}}\right).$$

If $S = \{1\}$, then

$$\delta_k := \delta_k(\{1\}) = \prod_p \left(1 - \frac{(p-1)^k}{p^k(p^k - 1)}\right).$$

Theorem 2.2 *If $k \geq 2$ and S is an arbitrary subset of* $\mathbf{N}$, *then the asymptotic densities of the sets of ordered k-tuples $(n_1, ..., n_k)$ such that $gcud(n_1, ..., n_k) \in S$ and $gcud(n_1, ..., n_k) = 1$ are $\delta_k(S)$ and δ_k, respectively, given in* Theorem 2.1.

Theorem 2.3 *Let p_n denote the n-th prime and for $r \in \mathbf{N}$ let $N = 10^r/2$. Then*

$$\delta_k \approx \prod_{n=1}^{N} \left(1 - \frac{(p_n - 1)^k}{p_n^k(p_n^k - 1)}\right)$$

is an approximation of δ_k with r exact decimals.

In particular, $\delta_2 \approx 0.8073, \delta_3 \approx 0.9637, \delta_4 \approx 0.9924, \delta_5 \approx 0.9983, \delta_6 \approx 0.9996, \delta_7 \approx 0.9999$, with $r = 4$ exact decimals.

Theorem 2.4 *For $k = 2$ the error term $R_2(x)$ of* (1.2) *can be improved into $R(x, S)$, where*

(i) *$R(x, S) = O(x \log x)$ for an S such that $\sum_{n \in S} \frac{1}{n} < \infty$ (in particular for every finite S) and for every multiplicative S,*

(ii) *$R(x, S) = O(x)$ for every multiplicative S such that $\sum_{p \notin S} \frac{1}{p} < \infty$ (in particular if the set $\{p : p \notin S\}$ is finite).*

Remark. It is noted in [1] that if $k = 2$ and if the function $\mu_S(n)$ is bounded, cf. proof of Theorem 2.4 of the present paper, then the error term is $R_2(x) = O(x \log x)$.

Theorem 2.5 *Suppose that $S \subseteq \mathbf{N}$ is multiplicative and $\min\{a : p^a \notin S\} \geq r \geq 2$ for every prime p. Then*

$$\sum_{n \leq x} \rho_S(n) = d(S)x + O(\sqrt[r]{x}), \tag{2.2}$$

where

$$d(S) = \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \sum_{\substack{a=1 \\ p^a \in S}}^{\infty} \frac{1}{p^a}\right) \tag{2.3}$$

is the asymptotic density of S.

Remark. In the special case $S =$ the set of K-void integers we reobtain from (2.2) the result of G. J. Rieger [7]. The K-void integers are defined as follows. Let K be a nonempty subset of $\mathbf{N} \setminus \{1\}$. The number n is called K-void if $n = 1$ or $n > 1$ and there is no prime power p^a, with $a \in K$, appearing in the canonical factorization of n.

Does the density exist for an arbitrary multiplicative subset S ? Yes, and it is $d(S)$ given by (2.3), where the infinite product is considered to be 0 when it diverges (if and only if $\sum_{p \notin S} \frac{1}{p} = \infty$). This follows from a well-known result of E. Wirsing [13] concerning the mean-values of certain multiplicative functions f. A short direct proof for the case f multiplicative and $0 \leq f(n) \leq 1$ for $n \geq 1$, hence applicable for the characteristic function of an arbitarary multiplicative S, is given in the book of G. Tenenbaum, [10], p. 48.

Theorem 2.6 *Let $k \geq 2$ and suppose that S is a completely multiplicative subset of* $\mathbf{N}$ *such that $\#\{n : n \in S \cap [1, x]\} = Ax + O(1)$. Then*

$$\#\{(n_1, ..., n_k) \in (S \cap [1, x])^k : gcud(n_1, ..., n_k) = 1\} = A^k \beta_k(S) x^k + T_k(x), \tag{2.4}$$

where

$$\beta_k(S)=\prod_{p\in S}\left(1-\frac{(p-1)^k}{p^k(p^k-1)}\right),$$

and $T_k(x)=O(x^{k-1})$ *for* $k>2$, $T_2(x)=O(x\log^2 x)$ *for* $k=2$.

If $Q_k^S(n)$ *denotes the probability that for* k *integers* $n_1,...,n_k$ *chosen at random from* $S\cap[1,n]$ *one has* $gcud(n_1,...,n_k)=1$*, then*

$$\lim_{n\to\infty} Q_k^S(n)=\beta_k(S).$$

3 Proofs

Proof of Theorem 2.1 Using the definition of μ_S^*, the fact that $d||gcud(n_1,...,n_k)$ if and only if $d||n_i$ for every $1\le i\le k$, which can be checked easily, and the well-known estimate

$$\sum_{\substack{n\le x\\ gcd(n,m)=1}} 1=\frac{\phi(m)x}{m}+O(\theta(m))$$

which holds uniformly for $x\ge 1$ and $m\in\mathbf{N}$, we obtain

$$\#\{(n_1,...,n_k)\in(\mathbf{N}\cap[1,x])^k : gcud(n_1,...,n_k)\in S\}=\sum_{n_1,...,n_k\le x}\rho_S(gcud(n_1,...,n_k))=$$

$$=\sum_{n_1,...,n_k\le x}\sum_{d||(n_1,...,n_k)}\mu_S^*(d)=\sum_{n_1,...,n_k\le x}\sum_{d||n_1,...,d||n_k}\mu_S^*(d)=$$

$$=\sum_{d\le x}\mu_S^*(d)\sum_{\substack{a_i\le x/d\\ (a_i,d)=1\\ 1\le i\le k}}1=\sum_{d\le x}\mu_S^*(d)\left(\sum_{\substack{a\le x/d\\ (a,d)=1}}1\right)^k=$$

$$=\sum_{d\le x}\mu_S^*(d)\left(\frac{x\phi(d)}{d^2}+O(\theta(d))\right)^k=\sum_{d\le x}\mu_S^*(d)\left(\frac{x^k\phi^k(d)}{d^{2k}}+O(\frac{x^{k-1}\theta(d)}{d^{k-1}})\right)=$$

$$=x^k\sum_{d\le x}\frac{\mu_S^*(d)\phi^k(d)}{d^{2k}}+O\left(x^{k-1}\sum_{d\le x}\frac{|\mu_S^*(d)|\theta(d)}{d^{k-1}}\right)=$$

$$=\delta_k(S)x^k+O\left(x^k\sum_{d>x}\frac{|\mu_S^*(d)|}{d^k}\right)+O\left(x^{k-1}\sum_{d\le x}\frac{|\mu_S^*(d)|\theta(d)}{d^{k-1}}\right).$$

The given error term yields now from the next statements:

(a)

$$\sum_{n\le x}\frac{\theta(n)}{n^s}=\begin{cases}O(\log^2 x), & s=1,\\ O(1), & s>1.\end{cases}$$

$$\sum_{n\le x}\frac{\theta^2(n)}{n^s}=\begin{cases}O(\log^4 x), & s=1,\\ O(1), & s>1,\end{cases}$$

$$\sum_{n>x}\frac{1}{n^s}=O(\frac{1}{x^{s-1}}),\qquad \sum_{n>x}\frac{\theta(n)}{n^s}=O(\frac{\log x}{x^{s-1}}),\quad s>1.$$

(b) For an arbitrary $S \subseteq \mathbf{N}$ and for every $n \in \mathbf{N}$, $|\mu^*_S(n)| \le \sum_{d||n} \rho_S(d) \le \theta(n)$, $|\mu^*_S(n)|\theta(n) \le \sum_{d||n} \rho_S(d)\theta(d)\theta(n/d)$ and

$$\sum_{n\le x} \frac{|\mu^*_S(n)|\theta(n)}{n} \le \sum_{d\le x} \frac{\rho_S(d)\theta(d)}{d} \sum_{e\le x/d} \frac{\theta(e)}{e} =$$

$$= O\left(\log^2 x \sum_{d\le x} \frac{\rho_S(d)\theta(d)}{d}\right) = \begin{cases} O(\log^2 x), & \text{if } \sum_{n=1}^{\infty} \frac{\rho_S(n)\theta(n)}{n} < \infty, \\ O(\log^4 x), & \text{otherwise.} \end{cases}$$

(c) If S is multiplicative, then μ^*_S is multiplicative too, $\mu^*_S(p^a) = \rho_S(p^a) - 1$ for every prime power p^a $(a \ge 1)$ and $\mu^*_S(n) \in \{-1, 0, 1\}$ for each $n \in \mathbf{N}$.

(d) Suppose S is multiplicative. Then

$$\sum_p \sum_{a=1}^{\infty} \frac{|\mu^*_S(p^a)|\theta(p^a)}{p^a} = 2 \sum_p \sum_{a=1}^{\infty} \frac{1-\rho_S(p^a)}{p^a} \le$$

$$\le 2 \sum_p \left(\frac{1-\rho_S(p)}{p} + \sum_{a=2}^{\infty} \frac{1}{p^a} \right) = 2 \sum_{p\in S} \frac{1}{p(p-1)} + 2 \sum_{p\notin S} \frac{1}{p-1} \le$$

$$\le 4 \left(\sum_{p\in S} \frac{1}{p^2} + \sum_{p\notin S} \frac{1}{p} \right) < \infty \quad \text{if} \quad \sum_{p\notin S} \frac{1}{p} < \infty.$$

It follows that in this case the series $\sum_{n=1}^{\infty} \frac{|\mu^*_S(n)|\theta(n)}{n}$ is convergent.

If S is multiplicative, then the series giving $\delta_k(S)$ can be expanded into an infinite product of Euler-type.

Proof of Theorem 2.2 This is a direct consequence of Theorem 2.1.

Proof of Theorem 2.3 Consider the series of positive terms

$$\sum_p \log \left(1 - \frac{(p-1)^k}{p^k(p^k-1)}\right)^{-1} =$$

$$= \sum_{n=1}^{\infty} \log \left(1 + \frac{(p_n-1)^k}{p_n^k(p_n^k-1) - (p_n-1)^k}\right) = -\log \delta_k,$$

where p_n denotes the n-th prime.

The N-th order error R_N of this series can be evaluated as follows:

$$R_N := \sum_{n=N+1}^{\infty} \log \left(1 + \frac{(p_n-1)^k}{p_n^k(p_n^k-1) - (p_n-1)^k}\right) < \sum_{n=N+1}^{\infty} \frac{(p_n-1)^k}{p_n^k(p_n^k-1) - (p_n-1)^k} <$$

$$< \sum_{n=N+1}^{\infty} \frac{1}{p_n^k - 1} \le \sum_{n=N+1}^{\infty} \frac{1}{p_n^2 - 1}.$$

Now using that $p_n > 2n$, valid for $n \ge 5$, we have

$$R_N < \sum_{n=N+1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2} \sum_{n=N+1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2(2N+1)}.$$

In order to obtain an approximation with r exact decimals we use the condition

$$\frac{1}{2(2N+1)} \leq \frac{1}{2} \cdot 10^{-r}$$

and obtain $N \geq \frac{1}{2}(10^r - 1)$.

The numerical values were obtained using the software package MAPLE.

Proof of Theorem 2.4 Define the function $\mu_S(n)$ by

$$\sum_{d|n} \mu_S(d) = \rho_S(n), \quad n \in \mathbf{N},$$

that is

$$\mu_S(n) = \sum_{d|n} \rho_S(d)\mu(n/d), \quad n \in \mathbf{N},$$

where $\mu(n) := \mu_{\{1\}}(n)$ is the Möbius function, see [1]. We have

$$N_k(x,S) := \#\{(n_1,...,n_k) \in (\mathbf{N} \cap [1,x])^k : gcd(n_1,...,n_k) \in S\} =$$

$$= \sum_{n_1,...,n_k \leq x} \rho_S(gcd(n_1,...,n_k)) = \sum_{n_1,...,n_k \leq x} \sum_{d|(n_1,...,n_k)} \mu_S(d)$$

and the proof runs parallel to the proof of Theorem 2.1.

Proof of Theorem 2.5

$$N_1(x,S) = \sum_{n \leq x} \rho_S(n) = \sum_{n \leq x} \sum_{d|n} \mu_S(d) = x \sum_{d \leq x} \frac{\mu_S(d)}{d} + O(\sum_{d \leq x} |\mu_S(d)|).$$

Here μ_S is multiplicative, $\mu_S(p^a) = \rho_S(p^a) - \rho_S(p^{a-1}), a \geq 1$ and since $p, p^2, ..., p^{r-1} \in S$ we have $\mu_S(p) = \mu_S(p^2) = ... = \mu_S(p^{r-1}) = 0$ for every prime p. Hence for each $n \in \mathbf{N}$, $|\mu_S(n)| \leq \rho_{L_r}(n)$, where L_r is the set of r-full numbers, i. e. $L_r = \{1\} \cup \{n > 1 : p|n \Rightarrow p^r|n\}$. We get

$$N_1(x,S) = d(S)x + O(x \sum_{d > x} \frac{\rho_{L_r}(d)}{d}) + O(\sum_{d \leq x} \rho_{L_r}(d)),$$

and using the elementary estimate

$$\sum_{n \leq x} \rho_{L_r}(n) = C\sqrt[r]{x} + O(\sqrt[r+1]{x}),$$

where C is a positive constant, due to P. Erdős and G. Szekeres [2], obtain the given result.

Proof of Theorem 2.6

$$\#\{(n_1,...,n_k) \in (S \cap [1,x])^k : gcud(n_1,...,n_k) = 1\} =$$

$$= \sum_{n_1 \leq x} \rho_S(n_1) ... \sum_{n_k \leq x} \rho_S(n_k) \sum_{d||(n_1,...,n_k)} \mu^*(d) = \sum_{n_1 \leq x} \rho_S(n_1) ... \sum_{n_k \leq x} \rho_S(n_k) \sum_{d||n_1,...,d||n_k} \mu^*(d) =$$

$$= \sum_{d \leq x} \mu^*(d) \sum_{\substack{a_1 \leq x/d \\ (a_1,d)=1}} \rho_S(da_1) ... \sum_{\substack{a_k \leq x/d \\ (a_k,d)=1}} \rho_S(da_k) =$$

$$= \sum_{d\le x} \rho_S(d)\mu^*(d) \left(\sum_{\substack{a\le x/d \\ (a,d)=1}} \rho_S(a) \right)^k .$$

Here we use the estimate, valid for every $\ell \in \mathbf{N}$,

$$\sum_{\substack{n\le x \\ gcd(n,\ell)=1}} \rho_S(n) = \sum_{n\le x} \rho_S(n) \sum_{d|gcd(n,\ell)} \mu(d) =$$

$$= \sum_{d|\ell} \mu(d)\rho_S(d) \sum_{e\le x/d} \rho_S(e) = \sum_{d|\ell} \mu(d)\rho_S(d) \left(A\frac{x}{d} + O(1) \right) =$$

$$= Ax \prod_{\substack{p|\ell \\ p\in S}} (1 - \frac{1}{p}) + O(\theta(\ell))$$

and obtain the desired result, see the proof of Theorem 2.1.

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