On a class of arithmetic functions

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1 Introduction

The aim of this paper is to establish an asymptotic formula with sharp remainder term for a class of multiplicative arithmetic functions, including certain special functions investigated in the literature.

2 Results

Theorem. Let $k \geq 2$ be a fixed integer and f be a multiplicative arithmetic function satisfying

(i) $f(p) = f(p^2) = ... = f(p^{k-1}) = 1$.

Suppose also that one of the following two conditions holds:

(ii) there exist $K > 0$ and $\beta > 0$ such that $|f(p^a)| \leq K p^{-\beta}$ for every $a \geq k$ and every prime p,

(ii') $f(p^k) = 0$ and there exist $M > 0$ and $\gamma < 1/k$ such that $|f(p^a)| \leq M p^{\gamma}$ for every $a > k$ and every prime p.

Then

$$\sum_{n \leq x} f(n) = C_f x + O(x^{1/k} \delta(x)), \tag{1}$$

where

$$C_f = \prod_p \left(1 + \sum_{a=k}^{\infty} \frac{f(p^a) - f(p^{a-1})}{p^a}\right)$$

and

$$\delta(x) = \exp(-A(\log x)^{3/5}(\log\log x)^{-1/5}),$$

where A is a positive constant.

Remarks. 1. By partial summation we obtain at once

$$\sum_{n\le x} nf(n) = \frac{1}{2}C_f x^2 + O(x^{1+1/k}\delta(x)), \tag{2}$$

which can be used in applications.

2. The slightly weaker error term $O(x^{1/k})$ can be obtained for (1) by elementary arguments, cf. the proof of the Theorem.

3 Applications

1. Let $f(n) = \gamma(n)/n$, where $\gamma(n)$ is the product of distinct prime factors of n (the square-free kernel or the core of n). Then $f(p^a) = p^{1-a}$ for every $a \ge 1$, therefore the conditions are verified with $k = 2$, $K = 1$, $\beta = 1$. Applying (2) we have

$$\sum_{n\le x} \gamma(n) = \frac{1}{2}C_1 x^2 + O(x^{3/2}\delta(x)), \tag{3}$$

where

$$C_1 = \prod_p \left(1 - \frac{1}{p(p+1)}\right).$$

Formula (3) is due (in a more general form) to D. Suryanarayana and P. Subrahmanyam [2] and improves the error term $O(x^{3/2})$, due to E. Cohen [1].

2. Let $f(n) = \tilde{\sigma}(n)/n$, where $\tilde{\sigma}$ is the multiplicative function given by

$$\tilde{\sigma}(p^a) = \sum_{\substack{1\le b\le a\\ (b,a)=1}} p^b$$

for every prime power p^a, $a \ge 1$. Then $f(p) = 1$, $f(p^a) \le \frac{1}{p^a}(p + p^2 + ... + p^{a-1}) < \frac{1}{p-1} \le \frac{2}{p}$ for every $a \ge 2$, hence the conditions of the Theorem are verified with $k = 2$, $K = 2$, $\beta = 1$ and we obtain from (2)

$$\sum_{n\le x} \tilde{\sigma}(n) = \frac{1}{2}C_2 x^2 + O(x^{3/2}\delta(x)), \tag{4}$$

where

$$C_2 = \prod_p \left(1 + \sum_{a=2}^{\infty} \frac{\tilde{\sigma}(p^a) - p\tilde{\sigma}(p^{a-1})}{p^{2a}}\right).$$

3. The positive integer $n = \prod p^a$ is called exponentially ℓ-free, where $\ell \ge 2$ is a fixed integer, if each exponent a in the canonical factorization of n is ℓ-free, i.e. each a is not divisible by the ℓ-th power of any prime. Let $q_\ell^{(e)}(n)$ denote the characteristic function of the exponentially ℓ-free integers.

Then $q_\ell^{(e)}(p) = q_\ell^{(e)}(p^2) = ... = q_\ell^{(e)}(p^{2^\ell-1}) = 1$, $q_\ell^{(e)}(p^{2^\ell}) = 0$, showing that the Theorem can be applied for $k = 2^\ell$, $M = 1$, $\gamma = 0$ and obtain from (1),

$$\sum_{n\le x} q_\ell^{(e)}(n) = C_3 x + O(x^{1/2^\ell}\delta(x)), \tag{5}$$

where

$$C_3 = \prod_p \left(1 + \sum_{a=2^\ell}^{\infty} \frac{q_\ell(p^a) - q_\ell(p^{a-1})}{p^a} \right),$$

where q_ℓ denotes the characteristic function of the ℓ-free integers.

In the special case $\ell = 2$ case formula (5) is due to J. Wu [5], improving an earlier result of M. V. Subbarao [3].

Note that our proof given in the next Section in more simple than the proof of J. Wu [5].

4 Proof of the Theorem

Write

$$F(s) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{\zeta(s)}{\zeta(ks)} V(s),$$

where

$$\frac{\zeta(s)}{\zeta(ks)} = \sum_{n=1}^{\infty} \frac{q_k(n)}{n^s}, \qquad V(s) := \sum_{n=1}^{\infty} \frac{v(n)}{n^s},$$

q_k denoting the characteristic function of the k-free integers.

Now the idea is to use the estimate

$$\sum_{n \le x} q_k(n) = \frac{1}{\zeta(k)} x + O(x^{1/k} \delta(x)), \tag{6}$$

of A. Walfisz [4], p. 192, formula (2). Note that using (6) with the slightly weaker error term $O(x^{1/k})$ which can be obtained by elementary arguments the error term of (1) will be also $O(x^{1/k})$.

We need to show that the absolute convergence abscissa of $V(s)$ is $< 1/k$.

In terms of the Dirichlet convolution we have $f = q_k * v$, $v = f * h_k * \mu$, where $h_k(n) = 1$ for $n = m^k$ and $h_k(n) = 0$ otherwise. Taking into account condition (i) we obtain for every $a = kb + c$, $0 < c < k$,

$$v(p^a) = f(p^a) - f(p^{a-1}) + f(p^{a-k}) - f(p^{a-k-1}) + ... + f(p^{k+c}) - f(p^{k+c-1}),$$

and for $a = kb$,

$$v(p^a) = f(p^a) - f(p^{a-1}) + f(p^{a-k}) - f(p^{a-k-1}) + ... + f(p^{2k}) - f(p^{2k-1}) + f(p^k),$$

important is that in all cases the smallest exponent appearing is $\ge k$.

Assuming condition (ii), we have $v(p^a) = 0$ for $1 \le a < k$ and for every $a \ge k$,

$$|v(p^a)| \le |f(p^a)| + |f(p^{a-1})| + ... + |f(p^k)| < \frac{aK}{p^\beta}$$

and for any positive s:

$$\sum_p \sum_{a=1}^{\infty} \frac{|v(p^a)|}{p^{as}} \ll \sum_p \sum_{a=k}^{\infty} \frac{a}{p^{as+\beta}} =$$

$$= \sum_p \frac{1}{p^{ks+\beta}} \left(k - \frac{k-1}{p^s}\right) \left(1 - \frac{1}{p^s}\right)^{-2}$$

which is convergent for $ks + \beta > 1$, i.e. for $s > \frac{1-\beta}{k}$. Therefore $V(s)$ is absolutely convergent for $s > \frac{1-\beta}{k}$, which is $< 1/k$.

Now assuming (ii'), we have $v(p^a) = 0$ for $1 \le a \le k$ and for every $a \ge k+1$,

$$|v(p^a)| \le |f(p^a)| + |f(p^{a-1})| + ... + |f(p^{k+1})| < aMp^\gamma,$$

and for any positive s,

$$\sum_p \sum_{a=1}^{\infty} \frac{|v(p^a)|}{p^{as}} \ll \sum_p \sum_{a=k+1}^{\infty} \frac{a}{p^{as-\gamma}} =$$

$$= \sum_p \frac{1}{p^{(k+1)s-\gamma}} \left(k + 1 - \frac{k}{p^s}\right) \left(1 - \frac{1}{p^s}\right)^{-2}$$

which is convergent for $(k+1)s - \gamma > 1$, i.e. for $s > \frac{1+\gamma}{k+1}$. Hence $V(s)$ is absolutely convergent for $s > \frac{1+\gamma}{k+1}$. This value is $< 1/k$, since $\gamma < 1/k$.

We have

$$\sum_{n \le x} f(n) = \sum_{d \le x} v(d) \sum_{e \le x/d} q_k(e) = \sum_{d \le x} v(d) \left(\frac{x}{\zeta(k)d} + O\left((\frac{x}{d})^k \delta(\frac{x}{d})\right)\right) =$$

$$= \frac{x}{\zeta(k)} \sum_{d \le x} \frac{v(d)}{d} + O\left(x^{1/k} \sum_{d \le x} \frac{|v(d)|}{d^{1/k}} \delta(\frac{x}{d})\right) =$$

$$= \frac{x}{\zeta(k)} V(1) + O\left(x \sum_{d > x} \frac{|v(d)|}{d}\right) + O\left(x^{1/k} \sum_{d \le x} \frac{|v(d)|}{d^{1/k}} (\frac{x}{d})^{-\varepsilon} (\frac{x}{d})^{\varepsilon} \delta(\frac{x}{d})\right),$$

where the first O-term is $O(x^u)$ with $u < 1/k$, by partial summation.

Furthermore, using that $x^\varepsilon \delta(x)$ is increasing for every $\varepsilon > 0$, the second O-term is

$$O\left(x^{1/k} x^\varepsilon \delta(x) x^{-\varepsilon} \sum_{d \le x} \frac{|v(d)|}{d^{1/k-\varepsilon}}\right) = O(x^{1/k} \delta(x))$$

for ε sufficiently small.

References

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