# ASYMPTOTIC FORMULAE CONCERNING ARITHMETICAL FUNCTIONS DEFINED BY CROSS-CONVOLUTIONS, II. THE DIVISOR FUNCTION

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*Studia Univ. Babeş-Bolyai, Math.*, **42** (1997), 105-110

**Abstract.** Let $A$ be a regular convolution of Narkiewicz and let $\tau_A(n)$ denote the number of $A$-divisors of $n$. We establish an asymptotic formula for the summatory function of $\tau_A$ if $A$ is a cross-convolution investigated in the first part of the present paper.

**Rezumat.– Formule asimptotice referitoare la funcţii aritmetice definite prin convoluţii "cross", II. Funcţia numărul divizorilor.** Fie $A$ o convoluţie regulată de tip Narkiewicz şi fie $\tau_A(n)$ numărul $A$-divizorilor lui $n$. În lucrare se stabileşte o formulă asimptotică pentru funcţia de sumare a lui $\tau_A$ în cazul în care $A$ este o convoluţie "cross" studiată de noi în prima parte a acestei lucrări.

*AMS Subject Classification*: 11A25, 11N37

*Key Words and Phrases*: Narkiewicz's regular convolution, divisor function, asymptotic formula

## 1. Introduction

In the first part [T97] of this paper and in [TH96] we introduced the notion of cross-convolution of arithmetical functions as a special case of Narkiewicz's [Nar63] regular convolution as follows. Let $\mathbb{N}$ denote the set of positive integers and let $A$ be a regular convolution given by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)g(n/d),$$

see [Nar63], [McC86], [Sit78], [T97]. The elements of the set $A(n)$ are called the $A$-divisors of $n$. We say that $A$ is a *cross-convolution* if for every prime $p$ we have either $A(p^a) = \{1, p, p^2, ..., p^a\} \equiv D(p^a)$ or $A(p^a) = \{1, p^a\} \equiv U(p^a)$ for every $a \in \mathbb{N}$. Let $P$ and $Q$ be the sets of the primes of the first and second kind of above, respectively, where $P \cup Q = \mathbb{P}$ is the set of all primes. For $P = \mathbb{P}$ and $Q = \emptyset$ we have the Dirichlet convolution $D$ and for $P = \emptyset$ and $Q = \mathbb{P}$ we obtain the unitary convolution $U$.

Furthermore, let $(P) = \{1\} \cup \{n \in \mathbb{N} :$ each prime factor of $n$ belongs to $P\}, (Q) = \{1\} \cup \{n \in \mathbf{N} :$ each prime factor of $n$ belongs to $Q\}$. Every $n \in \mathbb{N}$ can be written uniquely in the form $n = n_P n_Q$, where $n_P \in (P), n_Q \in (Q)$ and $(n_P, n_Q) = 1$. If $A$ is a cross-convolution, then $A(n) = \{d \in \mathbb{N} : d|n, (d, n/d) \in (P)\}$ for every $n \in \mathbb{N}$.

Let $\tau_A(n)$ denote the number of $A$-divisors of $n$. Observe that if $A$ is a cross-convolution, then $\tau_A(n) = \tau(n_P)\tau^*(n_Q) = \tau(n_P)2^{\omega(n_Q)}$, where $\omega(n_Q)$ represents the number of distinct prime factors of $n_Q$.

The aim of the present paper is to establish an asymptotic formula for the summatory function of $\tau_A$ if $A$ is a cross-convolution, which generalizes and unifies the corresponding known results concerning the divisor function $\tau$ and its unitary analogue $\tau^*$.

Our method is elementary and it applies an asymptotic estimate of B. Gordon and K. Rogers [GR64] concerning the divisor function $\tau$. The results of this paper are parts of our thesis [T95].

## 2. Preliminaries

We need the following lemmas.

**Lemma 1.** *Let $A$ be a cross-convolution and let $h$ be the multiplicative function defined by $h(1) = 1$ and*

$$h(p^a) = \begin{cases} -1, & \text{if } p \in Q \text{ and } a = 1, \\ 0, & \text{otherwise,} \end{cases}$$

*for every prime power $p^a$. Then*

$$\tau_A(n) = \sum_{d^2 e = n} h(d)\tau(e),$$

*for every $n \in \mathbb{N}$.*

*Proof.* Taking into account the multiplicativity of the functions $\tau_A, h$ and $\tau$ it is sufficient to verify the above identity for $n = p^a$, a prime power. Let $F(n) = \sum_{d^2 e = n} h(d)\tau(e)$. Then for $p \in P$ we have $F(p^a) = h(1)\tau(p^a) = \tau(p^a) = \tau_A(p^a)$, and for $p \in Q$ we obtain $F(p^a) = h(1)\tau(p^a) + h(p)\tau(p^{a-2}) = (a+1) - (a-1) = 2 = \tau^*(p^a) = \tau_A(p^a)$ if $a \geq 2$ and $F(p) = h(1)\tau(p) = 2 = \tau_A(p)$, which completes the proof.

If $Q = \mathbb{P}$, then $h$ is the Möbius function $\mu$ and we reobtain the identity given by M. V. Subbarao and D. Suryanarayana [SSur78], page 5.

The following properties were suggested by the paper of D. Suryanarayana [Sur69] and they represent generalizations of the results of that paper.

**Lemma 2.** *If $h$ is the function defined in Lemma 1, $u \in \mathbb{N}$ and $s > 1$, then*

$$\sum_{\substack{n=1\\(n,u)=1}}^{\infty} \frac{h(n)}{n^s} = \frac{u_Q^s}{\zeta_Q(s)\phi_s(u_Q)},$$

*the series being absolutely convergent, where*

$$\zeta_Q(s) = \sum_{\substack{n=1\\n\in(Q)}}^{\infty} \frac{1}{n^s} \qquad \text{and} \qquad \phi_s(n) = \sum_{d|n} d^s \mu(n/d).$$

*Proof.* The absolute convergence of the series follows at once by $|h(n)| \leq 1$ for every $n \in \mathbb{N}$ and by $s > 1$. Note that $\zeta_Q(s) = \prod_{p\in Q}(1 - \frac{1}{p^s})^{-1}$, where $s > 1$, see [T97], lemma 3. The function $h$ is multiplicative, hence we can use the Euler product formula:

$$\sum_{\substack{n=1\\(n,u)=1}}^{\infty} \frac{h(n)}{n^s} = \prod_{\substack{p\nmid u\\p\in Q}}(1 - \frac{1}{p^s}) = \prod_{p\in Q}(1 - \frac{1}{p^s}) \prod_{\substack{p|u\\p\in Q}}(1 - \frac{1}{p^s})^{-1} = \frac{u_Q^s}{\zeta_Q(s)\phi_s(u_Q)}.$$

**Lemma 3.** *If $h$ is the function defined in Lemma 1, $u \in \mathbb{N}$ and $s > 1$, then*

$$\sum_{\substack{n=1\\(n,u)=1}}^{\infty} \frac{h(n)\log n}{n^s} = \frac{u_Q^s}{\zeta_Q(s)\phi_s(u_Q)}(\alpha_s(u_Q) + \frac{\zeta_Q'(s)}{\zeta_Q(s)}),$$

*where $\zeta_Q'$ is the derivative of $\zeta_Q$ and*

$$\alpha_s(u) = \sum_{p|u} \frac{\log p}{p^s - 1}.$$

*Proof.* The series is uniformly convergent for $s \geq 1 + \varepsilon > 1$, with $\varepsilon > 0$ and the formula is obtained by termwise differentiation with respect to $s$ of the series given in Lemma 2, see [Sur69], lemma 2.5 for $h = \mu$.

We need the following result due by B. Gordon and K. Rogers [GR64], lemma 3, see also [SSur78], lemma 2.

**Lemma 4.** *If $u \in \mathbb{N}$, then*

$$\sum_{\substack{n\leq x\\(n,u)=1}} \tau(n) = (\frac{\phi(u)}{u})^2 x(\log x + 2C - 1 + 2\alpha(u)) + O(\sqrt{x}S(u)),$$

*where $\phi \equiv \phi_1$ is the Euler totient function, $C$ is Euler's constant and*

$$\alpha(u) \equiv \alpha_1(u) = \sum_{p|u} \frac{\log p}{p-1}, \qquad S(u) = \sum_{d|u} \frac{3^{\omega(d)}}{\sqrt{d}}.$$

**Lemma 5.** *If $A$ is a cross-convolution and the set $Q$ is finite, then for $s>0$ we have*

(i) $$\sum_{\substack{n\le x\\ n\in(Q)}} \frac{1}{n^s} = O(1),$$

(ii) $$\sum_{\substack{n>x\\ n\in(Q)}} \frac{1}{n^s} = O(\frac{1}{x^s}),$$

(iii) $$\sum_{\substack{n>x\\ n\in(Q)}} \frac{\log n}{n^s} = O(\frac{\log x}{x^s}).$$

*Proof.* For every $p\in Q$ define $\alpha_p\in\mathbb{N}$ such that $p^{\alpha_p}\le x<p^{\alpha_p+1}$. Then

$$\sum_{\substack{n\le x\\ n\in(Q)}} \frac{1}{n^s} \le \prod_{p\in Q}(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+...+\frac{1}{p^{\alpha_p s}}) < \prod_{p\in Q}(1-\frac{1}{p^s})^{-1},$$

which proves formula (i). Furthermore, let $Q=\{p_1,p_2,...,p_t\}$, where $p_1<p_2<...<p_t$. Then

$$\sum_{\substack{n>x\\ n\in(Q)}} \frac{1}{n^s} = \sum_{n=p_1^{a_1}...p_t^{a_t}>x} \frac{1}{(p_1^{a_1}...p_t^{a_t})^s} \le \sum_{2^{a_1+...+a_t}>x} \frac{1}{2^{s(a_1+...+a_t)}},$$

where $p_1^{a_1}...p_t^{a_t}\ge p_1^{a_1+...+a_t}\ge 2^{a_1+...+a_t}$ and $2^{a_1+...+a_t}>x$ implies $p_1^{a_1}...p_t^{a_t}>x$.

Denoting $a_1+...+a_t=a\in\mathbb{N}$ and $a(x)=\log x/\log 2$ we have

$$\sum_{\substack{n>x\\ n\in(Q)}} \frac{1}{n^s} \le \sum_{2^a>x}\frac{1}{2^{sa}} = \sum_{a>a(x)}\frac{1}{2^{sa}} = \frac{1}{(2^s-1)2^{s[a(x)]}} = O(\frac{1}{x^s}),$$

where $[a(x)]$ stands for the integer part of $a(x)$, and it follows (ii). Formula (iii) can be proved in the same way.

## 3. Main results

Now we are ready to prove the following asymptotic formula.

**Theorem 1.** *If $A$ is a cross-convolution and $u\in\mathbb{N}$, then*

$$\sum_{\substack{n\le x\\ (n,u)=1}} \tau_A(n) = (\frac{\phi(u)}{u})^2\frac{u_Q^2 x}{\zeta_Q(2)\phi_2(u_Q)}(\log x+2C-1+2\alpha(u)-2\beta(u_Q)-2\frac{\zeta_Q'(2)}{\zeta_Q(2)})$$

$$+O(H(x,Q)S(u)),$$

*where*

$$\beta(u) \equiv \alpha_2(u) = \sum_{p|u} \frac{\log p}{p^2-1},$$

*and* $H(x,Q) = \sqrt{x}$ *(*$Q$ *finite),* $\sqrt{x}\log x$ *(*$Q$ *infinite).*

*Proof.* Using Lemmas 1 and 4 we deduce

$$\sum_{\substack{n\le x\\(n,u)=1}} \tau_A(n) = \sum_{\substack{d^2e=n\le x\\(d^2e,u)=1}} h(d)\tau(e) = \sum_{\substack{d\le\sqrt{x}\\(d,u)=1}} h(d) \sum_{\substack{e\le x/d^2\\(e,u)=1}} \tau(e)$$

$$= \sum_{\substack{d\le\sqrt{x}\\(d,u)=1}} h(d)\left( (\frac{\phi(u)}{u})^2 \frac{x}{d^2}(\log\frac{x}{d^2} + 2C - 1 + 2\alpha(u)) + O(S(u)\sqrt{\frac{x}{d^2}})\right)$$

$$= (\frac{\phi(u)}{u})^2 x(\log x + 2C - 1 + 2\alpha(u)) \sum_{\substack{d\le\sqrt{x}\\(d,u)=1}} \frac{h(d)}{d^2}$$

$$-2x(\frac{\phi(u)}{u})^2 \sum_{\substack{d\le\sqrt{x}\\(d,u)=1}} \frac{h(d)\log d}{d^2} + O(S(u)\sqrt{x} \sum_{\substack{d\le\sqrt{x}\\d\in(Q)}} \frac{1}{d}).$$

Applying Lemmas 2 and 3 for $s=2$ and the well-known estimates

$$\sum_{n>x} \frac{1}{n^s} = O(x^{1-s}), \quad s>1,$$
$$\sum_{n>x} \frac{\log n}{n^s} = O(x^{1-s}\log x), \quad s>1.$$

we have

$$\sum_{\substack{d\le\sqrt{x}\\(d,u)=1}} \frac{h(d)}{d^2} = \sum_{\substack{d=1\\(d,u)=1}}^{\infty} \frac{h(d)}{d^2} + O(\sum_{\substack{d>\sqrt{x}\\d\in(Q)}} \frac{1}{d^2})$$

$$= \frac{u_Q^2}{\zeta_Q(2)\phi_2(u_Q)} + O(\frac{1}{(\sqrt{x})^m}),$$

$$\sum_{\substack{d\le\sqrt{x}\\(d,u)=1}} \frac{h(d)\log d}{d^2} = \sum_{\substack{d=1\\(d,u)=1}}^{\infty} \frac{h(d)\log d}{d^2} + O(\sum_{\substack{d>\sqrt{x}\\d\in(Q)}} \frac{\log d}{d^2})$$

$$= \frac{u_Q^2}{\zeta_Q(2)\phi_2(u_Q)}(\beta(u_Q) + \frac{\zeta_Q'(2)}{\zeta_Q(2)}) + O(\frac{\log x}{(\sqrt{x})^m}),$$

where $m = 2$ if $Q$ is finite, see Lemma 5 and $m = 1$ if $Q$ is infinite. We also have

$$\sum_{\substack{n \le x \\ n \in (Q)}} \frac{1}{n} = \begin{cases} O(1), & \text{if } Q \text{ is finite,} \\ O(\log x), & \text{otherwise,} \end{cases}$$

see Lemma 5/(i), which completes the proof.

In the unitary case we reobtain the formula proved by M. V. Subbarao and D. Suryanarayana [SSur78], lemma 3 for the function $\tau^*$.

For $u = 1$ we obtain the following result.

**Theorem 2.** *If $A$ is a cross-convolution, then*

$$\sum_{n \le x} \tau_A(n) = \frac{x}{\zeta_Q(2)} (\log x + 2C - 1 - 2\frac{\zeta_Q'(2)}{\zeta_Q(2)}) + O(H(x, Q)),$$

*where $H(x, Q)$ is defined by Theorem 1.*

For $A = D$ we reobtain Dirichlet's well-known formula and for $A = U$ we have the formula established by Mertens, see E. Cohen [Co60].

The remainder term the above formula can be improved for $A = U$ into $O(\sqrt{x})$ using analytical methods, see A. A. Gioia and A. M. Vaidya [GV66].

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