Asymptotic properties of functions defined on arithmetic semigroups

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1 Introduction

Asymptotic properties of certain special arithmetical functions, like the divisor function $\tau(n)$, the sum-of-divisors function $\sigma(n)$ and the Euler function $\phi(n)$ can be investigated in the general setting of arithmetical semigroups. The book of J. Knopfmacher [8] contains a detailed study of such properties, including asymptotic formulae with error estimates and extremal orders of magnitude of functions.

This book gives also properties of the unitary analogue $\tau^*(n)$ of the function $\tau(n)$, while the other unitary analogues $\sigma^*(n)$ and $\phi^*(n)$, cf. E. Cohen [2], [3], P. J. McCarthy [10], are not investigated. We refer to the papers of E. M. Horadam [6], [7] for the study of these functions defined for Beurling's generalized integers, which is essentially the same as the setting of arithmetical semigroups.

Common generalizations of these functions are given in terms of regular convolutions of arithmetical functions, cf. W.Narkiewicz [11] and [12], [10], [13].

In a series of papers, see [13], [14] we gave asymptotic formulae for functions defined on the set $\mathbf{N}$ of positive integers by a special case of regular convolutions, called cross-convolutions. These results generalize and unify the corresponding known results concerning the usual and the unitary functions.

In the present paper we show that our results can be extended for functions defined on arithmetical semigroups satisfying the basic axiom.

We will refer to the concepts and results given in the book [8]. We summarize in Sections 2-4 the notions and results we need regarding arithmetical semigroups and regular convolutions. Our results are given in Sections 5-8.

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2 Arithmetical semigroups

Let $(G, \cdot)$ be a commutative semigroup with identity $\mathbf{1}$ and suppose that there exists a finite or countable infinite subset $P(G) \equiv P$ of G such that every $\mathbf{n} \in G$ has a unique factorization, up to the order of the factors, of the form

$$\mathbf{n} = \prod_{\mathbf{p}\in P} \mathbf{p}^{\mathbf{n}(\mathbf{p})},$$

where the exponents $\mathbf{n}(\mathbf{p})$ are non-negative integers of which all but a finite number are zero (define $\mathbf{1}(\mathbf{p}) = 0$ for all $\mathbf{p} \in P$). Furthermore, suppose that there exists a norm mapping $|\cdot| : G \to \mathbf{R}$ such that

(i) $|\mathbf{1}| = 1, |\mathbf{p}| > 1$ for each $\mathbf{p} \in P$,

(ii) $|\mathbf{mn}| = |\mathbf{m}||\mathbf{n}|$ for each $\mathbf{m}, \mathbf{n} \in G$,

(iii) $N_G(x) \equiv \#\{\mathbf{n} \in G : |\mathbf{n}| \le x\}$ is finite for every real $x > 0$.

Then G is called an *arithmetical semigroup*. The elements of G are called *generalized integers* (typed in boldface). The elements of P are the *primes*.

For $\mathbf{d}, \mathbf{n} \in G$, $\mathbf{d}$ is said to be a *divisor* of $\mathbf{n}$ if $\mathbf{de} = \mathbf{n}$ for some $\mathbf{e} \in G$, notation $\mathbf{d}|\mathbf{n}$. The *greatest common divisor* $(\mathbf{m}, \mathbf{n})$ and the least common multiple $[\mathbf{m}, \mathbf{n}]$ of $\mathbf{m}, \mathbf{n} \in G$ are defined in the usual way. We have

(a) $\mathbf{d}|\mathbf{n}$ if and only if $\mathbf{d}(\mathbf{p}) \le \mathbf{n}(\mathbf{p})$ for every $\mathbf{p} \in P$,

(b) $(\mathbf{m}, \mathbf{n})(\mathbf{p}) = \min\{\mathbf{m}(\mathbf{p}), \mathbf{n}(\mathbf{p})\}$ for every $\mathbf{p} \in P$,

(c) $[\mathbf{m}, \mathbf{n}](\mathbf{p}) = \max\{\mathbf{m}(\mathbf{p}), \mathbf{n}(\mathbf{p})\}$ for every $\mathbf{p} \in P$.

G satisfies the *basic axiom* (Axiom A, cf. [8], p. 75) if there exist real constants Δ, δ and η such that $\Delta > 0$, $0 \le \eta < \delta$ and

$$N_G(x) = \Delta x^\delta + O(x^\eta) \qquad \text{as} \quad x \to \infty.$$

If G satisfies the basic axiom, then the set P of primes is infinite and has asymptotic density zero, i. e. $\pi_G(x) \equiv \#\{\mathbf{p} \in P : |\mathbf{p}| \le x\} = o(N_G(x))$ as $x \to \infty$ (cf. [8], p. 87).

Complex valued functions defined on G are called *arithmetical functions*. The arithmetical function f is *multiplicative* (*completely multiplicative*) if $f(\mathbf{mn}) = f(\mathbf{m})f(\mathbf{n})$ for every coprime $\mathbf{m}, \mathbf{n} \in G$, i. e. with $(\mathbf{m}, \mathbf{n}) = \mathbf{1}$ (for every $\mathbf{m}, \mathbf{n} \in G$).

A "normalization principle" (see [8], p. 78): Estimates regarding sums of the form $\sum_{|\mathbf{n}|\le x} f(\mathbf{n})$, where f is an arithmetic function, may be simplified considering the norm $|\mathbf{n}|^* = |\mathbf{n}|^\delta$. Then

$$\sum_{|\mathbf{n}|^* \le x} 1 = \sum_{|\mathbf{n}|^\delta \le x} 1 = N_G(x^{1/\delta}) = \Delta x + O(x^{\eta/\delta})$$

and the basic axiom can be stated as follows: there exist real constants Δ and η such that $\Delta > 0$, $0 \le \eta < 1$ and

$$\sum_{|\mathbf{n}|^* \le x} 1 = \Delta x + O(x^\eta).$$

That is, this converts G into an arithmetic semigroup G^* satisfying the basic axiom with $\delta = 1$. We do not use it in what follows.

Examples: 1) The set $G = \mathbf{N}$ of positive integers forms an arithmetical semigroup with respect to ordinary multiplication and the norm $|n| = n, n \in \mathbf{N}$. Here $P(\mathbf{N})$ is the set of all rational primes and since

$$N_{\mathbf{N}}(x) = [x] = x + O(1),$$

the basic axiom is satisfied with $\Delta = \delta = 1, \eta = 0$. This is the proto-type of all arithmetical semigroups.

2) Let $(p_n)_{n\ge1}$ be an infinite sequence of real numbers such that $1 < p_1 < p_2 < ... < p_n < ...$. Then the numbers $p_1^{a_1} p_2^{a_2} ...$, where the exponents are non-negative integers of which all

but a finite number are zero, are called Beurling's generalized integers, cf. A. BEURLING [1]. They form an arithmetical semigroup with respect to ordinary multiplication and the norm $|n| = n$. The basic axiom may or may not be satisfied.

3) If $(D, +, \cdot)$ is an Euclidean domain, then the set G_D of all associate classes $\bar{a}$ of non-zero elements $a \in D$ forms a commutative semigroup with identity under the multiplication operation $\overline{a} \cdot \overline{b} = \overline{ab}$. Furthermore, we have a unique factorization into powers of the classes $\overline{p}$ of prime elements $p \in D$ and $|\overline{a}| = |a|$ defines a norm on G_D verifying conditions (i) and (ii) of above. In certain special cases condition (iii) is satisfied too and G_D forms an arithmetical semigroup.

For example, if $D = \mathbf{Z}$ the ring of rational integers, then G_Z can be identified with the semigroup $\mathbf{N}$ of Example 1.

Let $D = \mathbf{Z}[i] = \{m + ni : m, n \in \mathbf{Z}\}$ be the set of the Gaussian integers. $(\mathbf{Z}[i], +, \cdot)$ is an Euclidean domain with the norm $|m + ni| = m^2 + n^2$, it has 4 units (the numbers $1, -1, i$ and $-i$), hence

$$N_{G_{\mathbf{Z}[i]}}(x) \equiv N_{\mathbf{Z}[i]}(x) = \frac{1}{4} \sum_{n \le x} r(n) < \infty,$$

where $r(n) = \#\{(x, y) \in \mathbf{Z}^2 : x^2 + y^2 = n\}$, i. e. $r(n)$ is the number of lattice points on the circle $x^2 + y^2 = n$ $(n \ge 1)$. Hence $G_{\mathbf{Z}[i]}$ is an arithmetical semigroup. Here the primes are the associate classes of the numbers

(i) $1 + i$,

(ii) the rational primes $p \equiv 3 \pmod 4$,

(iii) the factors of the rational primes $p \equiv 1 \pmod 4$.

We also have

$$N_{\mathbf{Z}[i]}(x) = \frac{\pi}{4} x + O(x^{1/2}),$$

therefore the basic axiom is satisfied with $\Delta = \frac{\pi}{4}, \delta = 1, \eta = \frac{1}{2}$. Note that the constant η of the basic axiom is not unique. The above estimate with $\eta = \frac{1}{2}$ is due to C. F. GAUSS and it can be obtained by simple familiar arguments. According to a result of H. IWANIEC and C. J. MAZZOCHI (1987) one can choose $\eta = \frac{7}{22} + \varepsilon$ for every $\varepsilon > 0$, where $\frac{7}{22} \approx 0.318181$. On the other hand $\eta \ge \frac{1}{4} = 0.25$, as it was shown by E. LANDAU (1915) and independently by G. H. HARDY (1915), see [9], p. 141-142.

4) Let K denote an algebraic number field, i. e. a finite extension of the rational field $\mathbf{Q}$ and let D denote the ring of the algebraic integers in K. For example, if $K = \mathbf{Q}$ and $K = \mathbf{Q}(i)$, then we have the rings $\mathbf{Z}$ and $\mathbf{Z}[i]$, respectively. Now, the set G_K of the non-zero ideals in D (the integral ideals in K) forms an arithmetical semigroup satisfying the basic axiom, see [8], p. 14 and p. 76.

3 Asymptotic estimates for semigroups satisfying the basic axiom

Suppose that G is an arithmetical semigroup satisfying the basic axiom. The first result of this Section is not given in [8], see [5], Lemma 1.

Theorem 3.1 *Suppose that f is a multiplicative function and the series $\sum_{\mathbf{n} \in G} f(\mathbf{n})$ is absolutely convergent. Then*

$$\sum_{\mathbf{n} \in G} f(\mathbf{n}) = \prod_{\mathbf{p} \in P} \sum_{a=0}^{\infty} f(\mathbf{p}^a),$$

i. e. the Euler product formula holds and the product is also absolutely convergent.

Theorem 3.2 *The series* $\sum_{\mathbf{n}\in G} |\mathbf{n}|^{-z}$ *is absolutely convergent for* $\operatorname{Re} z > \delta$ *and divergent for* $\operatorname{Re} z \le \delta$*. Hence*

$$\zeta_G(z) \equiv \zeta(z) \equiv \sum_{\mathbf{n}\in G} |\mathbf{n}|^{-z} = \prod_{\mathbf{p}\in P} \left(1 - \frac{1}{|\mathbf{p}|^z}\right)^{-1}, \qquad \operatorname{Re} z > \delta$$

is an analytic function.

$\zeta_G(z)$ is called the *zeta function of the arithmetical semigroup* G.

In what follows we give estimates for the sum $\sum_{|\mathbf{n}|\le x} |\mathbf{n}|^s$, where $s \in \mathbf{R}$. For the proof we refer once again to [8], Ch. 4.

Theorem 3.3

(i)
$$\sum_{|\mathbf{n}|\le x} |\mathbf{n}|^{-s} = \zeta_G(s) + O(x^{-s+\delta}), \quad s > \delta,$$

(ii)
$$\sum_{|\mathbf{n}|\le x} |\mathbf{n}|^{-\delta} = \Delta\delta \log x + \gamma_G + O(x^{-\delta+\eta}),$$

where $\gamma_G = \lim_{s\to\delta} \left(\zeta_G(s) - \frac{\Delta\delta}{s-\delta}\right)$ *is the Euler constant of* G*,*

(iii)
$$\sum_{|\mathbf{n}|\le x} |\mathbf{n}|^{-\eta} = \frac{\Delta\delta}{\delta-\eta} x^{\delta-\eta} + O(\log x),$$

(iv)
$$\sum_{|\mathbf{n}|\le x} |\mathbf{n}|^{-s} = \frac{\Delta\delta}{\delta-s} x^{\delta-s} + \alpha(s) + O(x^{-s+\eta}), \quad \eta < s < \delta,$$

where $\alpha(s)$ *is a constant, depending on* s*,*

(v)
$$\sum_{|\mathbf{n}|\le x} |\mathbf{n}|^{s} = \frac{\Delta\delta}{s+\delta} x^{s+\delta} + O(x^{s+\eta}), \quad s > -\eta.$$

Theorem 3.4 *(Mertens' formula, see [8], p. 171)*

$$\prod_{|\mathbf{p}|\le x} \left(1 - \frac{1}{|\mathbf{p}|^{\delta}}\right) = \frac{e^{-\gamma}}{\delta\Delta \log x}\left(1 + O\left(\frac{1}{\log x}\right)\right),$$

where $\gamma \approx 0.5772$ *is the Euler-constant (of* $\mathbf{N}$*).*

4 Regular convolutions

Let $A(\mathbf{n})$ be a subset of the set of divisors of $\mathbf{n}$ for each $\mathbf{n} \in G$. The A-convolution of the arithmetical functions f and g is given by

$$(f *_A g)(\mathbf{n}) = \sum_{\mathbf{d} \in A(\mathbf{n})} f(\mathbf{d}) g(\mathbf{n}/\mathbf{d}).$$

This convolution is called *regular* if

(a) the set of arithmetical functions is a commutative ring with identity with respect to ordinary addition and the A-convolution,

(b) the A-convolution of multiplicative functions is multiplicative,

(c) the function I, defined by $I(\mathbf{n}) = 1, \mathbf{n} \in G$, has an inverse $\mu_{G,A} \equiv \mu_A$ with respect to the A-convolution and $\mu_A(\mathbf{p}^a) \in \{-1, 0\}$ for every prime power $\mathbf{p}^a$ with $a \geq 1$.

For $A = D$, $\mu_D \equiv \mu$ is the Möbius function given by

$$\mu(\mathbf{n}) = \begin{cases} 1, & \text{if } \mathbf{n} = \mathbf{1}, \\ (-1)^r, & \text{if } \mathbf{n} = \mathbf{p}_1 \mathbf{p}_2 \cdots \mathbf{p}_r, \quad \mathbf{p}_i \in P \text{ distinct}, \\ 0, & \text{otherwise}, \end{cases}$$

Regular convolutions were introduced by W. Narkiewicz [11] in the case $G = \mathbf{N}$ and for an arithmetical semigroup by P. Haukkanen [4].

For example, the Dirichlet convolution D, where $D(\mathbf{n}) = \{\mathbf{d} \in G : \mathbf{d}|\mathbf{n}\}$, and the unitary convolution U, where $U(\mathbf{n}) = \{\mathbf{d} \in G : \mathbf{d}|\mathbf{n}, (\mathbf{d}, \mathbf{n}/\mathbf{d}) = \mathbf{1}\}$, are regular.

It can be proved, see [11] that an A-convolution is regular if and only if

(i) $A(\mathbf{mn}) = \{\mathbf{de} : \mathbf{d} \in A(\mathbf{m}), \mathbf{e} \in A(\mathbf{n})\}$ for every $\mathbf{m}, \mathbf{n} \in G, (\mathbf{m}, \mathbf{n}) = \mathbf{1}$,

(ii) for every prime power $\mathbf{p}^a, a \geq 1$ there exists a divisor $t = t_A(\mathbf{p}^a)$ of a, called the type of $\mathbf{p}^a$ with respect to A, such that $A(\mathbf{p}^{it}) = \{\mathbf{1}, \mathbf{p}^t, \mathbf{p}^{2t}, ..., \mathbf{p}^{it}\}$ for every $i \in \{0, 1, ..., a/t\}$.

The elements of the set $A(\mathbf{n})$ are called the A-divisors of $\mathbf{n}$.

For other properties of regular convolutions see also [10] and [12].

We say that the regular convolution A is a *cross-convolution* if for every prime $\mathbf{p}$ we have either $t_A(\mathbf{p}^a) = 1$, i.e. $A(\mathbf{p}^a) = \{\mathbf{1}, \mathbf{p}, \mathbf{p}^2, ..., \mathbf{p}^a\} \equiv D(\mathbf{p}^a)$ for every $a \in \mathbf{N}$ or $t_A(\mathbf{p}^a) = a$, i.e. $A(\mathbf{p}^a) = \{\mathbf{1}, \mathbf{p}^a\} \equiv U(\mathbf{p}^a)$ for every $a \in \mathbf{N}$. Let P_A and Q_A be the sets of the primes of the first and second kind of above, respectively, where $P_A \cup Q_A = P$ is the set of all primes. For $P_A = P$ and $Q = \emptyset$ we have the Dirichlet convolution D and for $P = \emptyset$ and $Q = P$ we obtain the unitary convolution U.

Furthermore, let (P_A) and (Q_A) denote the multiplicative semigroups generated by $\{\mathbf{1}\} \cup P_A$ and $\{\mathbf{1}\} \cup Q_A$, respectively. Every $\mathbf{n} \in G$ can be written uniquely in the form $\mathbf{n} = \mathbf{n}_{P_A} \mathbf{n}_{Q_A}$, where $\mathbf{n}_{P_A} \in (P_A), \mathbf{n}_{Q_A} \in (Q_A)$.

If A is a cross-convolution, then

$$A(\mathbf{n}) = \{\mathbf{d} \in G : \mathbf{d}|\mathbf{n}, (\mathbf{d}, \mathbf{n}/\mathbf{d}) \in (P_A)\}$$

and we get

$$(f *_A g)(\mathbf{n}) = \sum_{\substack{\mathbf{d}|\mathbf{n} \\ (\mathbf{d}, \mathbf{n}/\mathbf{d}) \in (P_A)}} f(\mathbf{d}) g(\mathbf{n}/\mathbf{d}).$$

If A is a cross-convolution, let

$$\zeta_{P_A}(z) \equiv \sum_{n \in (P_A)} |\mathbf{n}|^{-z} = \prod_{\mathbf{p} \in P_A} \left(1 - \frac{1}{|\mathbf{p}|^z}\right)^{-1},$$

$$\zeta_{Q_A}(z) \equiv \sum_{n\in(Q_A)} |\mathbf{n}|^{-z} = \prod_{\mathbf{p}\in Q_A} \left(1-\frac{1}{|\mathbf{p}|^z}\right)^{-1},$$

where $\zeta_{P_A}(z)\zeta_{Q_A}(z) = \zeta(z)$, $\operatorname{Re} z > \delta$.

Cross-convolutions were introduced by us in [13] in case $G = \mathbf{N}$.

5 Arithmetical functions

Consider the following functions, where A is a regular convolution and $s \in \mathbf{C}$:

$\sigma_{G,A,s}(\mathbf{n}) \equiv \sigma_{A,s}(\mathbf{n}) = \sum_{\mathbf{d}\in A(\mathbf{n})} |\mathbf{d}|^s =$ the sum of s-th powers of the norms of A-divisors of $\mathbf{n}$, $\sigma_{A,1}(\mathbf{n}) \equiv \sigma_A(\mathbf{n})$, $\sigma_{A,0}(\mathbf{n}) = \tau_A(\mathbf{n}) =$ the number of A-divisors of $\mathbf{n}$.

Furthermore, define the following generalizations of the Euler function:

$$\phi_{G,A}(x,\mathbf{n},s) \equiv \phi_A(x,\mathbf{n},s) = \sum_{\substack{|\mathbf{k}|\le x \\ (\mathbf{k},\mathbf{n})_A=\mathbf{1}}} |\mathbf{k}|^s,$$

where $x > 0$ and the sum is over the elements $\mathbf{k} \in G$ such that $|\mathbf{k}| \le x$ and $(\mathbf{k},\mathbf{n})_A = \mathbf{1}$,

$$\phi_A(|\mathbf{n}|,\mathbf{n},0) \equiv \phi_A(\mathbf{n}) = \sum_{\substack{|\mathbf{k}|\le|\mathbf{n}| \\ (\mathbf{k},\mathbf{n})_A=\mathbf{1}}} 1,$$

representing the number of elements $\mathbf{k} \in G$ such that $|\mathbf{k}| \le |\mathbf{n}|$ and $(\mathbf{k},\mathbf{n})_A = \mathbf{1}$, this is given in [8], p. 40 for $G = \mathbf{N}$,

$$\varphi_{A,s}(\mathbf{n}) = \sum_{\mathbf{d}\in A(\mathbf{n})} \mu_A(\mathbf{d})|\mathbf{n}/\mathbf{d}|^s,$$

$$\varphi_{A,\delta}(\mathbf{n}) = \sum_{\mathbf{d}\in A(\mathbf{n})} \mu_A(\mathbf{d})|\mathbf{n}/\mathbf{d}|^\delta.$$

Note that for $G = \mathbf{N}$, $\phi_A(\mathbf{n}) \equiv \varphi_{A,\delta}(\mathbf{n})$, which gives in case $A = D$ the classical Euler function. In an arbitrary semigroup G, $\phi_A(\mathbf{n})$ and $\varphi_{A,\delta}(\mathbf{n})$ are distinct.

Theorem 5.1 *If G is an arithmetical semigroup (not necessarily verifying the basic axiom) and A is a regular convolution, then*

i) the function $\sigma_{A,s}$ is multiplicative and

$$\sigma_{A,s}(\mathbf{n}) = \prod_{\mathbf{p}\in P} \frac{|\mathbf{p}|^{(\mathbf{n}(\mathbf{p})+t)s}-1}{|\mathbf{p}|^{st}-1}, \quad t = t_A(\mathbf{p}^{\mathbf{n}(\mathbf{p})}), \qquad s \ne 0,$$

$$\tau_A(\mathbf{n}) = \prod_{\mathbf{p}\in P} (\mathbf{n}(\mathbf{p})+1).$$

ii)

$$\phi_A(x,\mathbf{n},s) = \sum_{\mathbf{d}\in A(\mathbf{n})} \mu_A(\mathbf{d})|\mathbf{d}|^s \sum_{|\mathbf{e}|\le \frac{x}{|\mathbf{d}|}} |\mathbf{e}|^s,$$

$$\phi_A(\mathbf{n}) = \sum_{\mathbf{d}\in A(\mathbf{n})} \mu_A(\mathbf{d}) N_G(|\mathbf{n}/\mathbf{d}|).$$

iii) the function $\varphi_{A,s}$, in particular $\varphi_{A,\delta}$, is multiplicative and

$$\varphi_{A,s}(\mathbf{n}) = |\mathbf{n}|^s \prod_{\substack{\mathbf{p}\in P \\ \mathbf{n}(\mathbf{p})\geq 1}} \left(1-|\mathbf{p}|^{-ts}\right),$$

$$\varphi_{A,\delta}(\mathbf{n}) = |\mathbf{n}|^\delta \prod_{\substack{\mathbf{p}\in P \\ \mathbf{n}(\mathbf{p})\geq 1}} \left(1-|\mathbf{p}|^{-t\delta}\right), \qquad t = t_A(\mathbf{p}^{\mathbf{n}(\mathbf{p})}).$$

Proof. Similarly to case $G = \mathbf{N}$. For ii) use that $\mathbf{d} \in A((\mathbf{k},\mathbf{n})_A)$ if and only if $\mathbf{d}|\mathbf{k}$ and $\mathbf{d} \in A(\mathbf{n})$, see [12], Theorem 4.2,

$$\phi_A(x,\mathbf{n},s) = \sum_{|\mathbf{k}|\leq x} |\mathbf{k}|^s \sum_{\mathbf{d}\in A((\mathbf{k},\mathbf{n})_A)} \mu_A(\mathbf{d}) = \sum_{|\mathbf{k}|\leq x} |\mathbf{k}|^s \sum_{\substack{\mathbf{d}|\mathbf{k} \\ \mathbf{d}\in A(\mathbf{n})}} \mu_A(\mathbf{d}) =$$

$$= \sum_{\mathbf{d}\in A(\mathbf{n})} \mu_A(\mathbf{d})|\mathbf{d}|^s \sum_{|\mathbf{e}|\leq \frac{x}{|\mathbf{d}|}} |\mathbf{e}|^s,$$

where $\mathbf{k} = \mathbf{de}$.

Remark. For $s \geq 0$ we have $\varphi_{A,s}(\mathbf{n}) \leq |\mathbf{n}|^s$ for every $\mathbf{n} \in G$.

6 Asymptotic estimates for divisor-sum functions and Euler-type functions defined on arithmetical semigroups

Theorem 6.1 *If G satisfies the basic axiom, A is a regular convolution, $\mathbf{n} \in G$, $s > -\eta$ and $0 \leq \varepsilon < \delta - \eta$, then*

(a)
$$\phi_A(x,\mathbf{n},s) = \frac{\Delta\delta}{s+\delta} \cdot \frac{\varphi_{A,\delta}(\mathbf{n})}{|\mathbf{n}|^\delta} x^{s+\delta} + O(x^{s+\eta+\varepsilon}\sigma_{-\eta-\varepsilon}(\mathbf{n})),$$

where $\sigma_w(\mathbf{n}) \equiv \sigma_{D,w}(\mathbf{n})$ is the sum of w-th powers of the norms of divisors of $\mathbf{n}$,

(b)
$$\phi_A(\mathbf{n}) = \Delta\varphi_{A,\delta}(\mathbf{n}) + O(|\mathbf{n}|^\eta \sigma_{-\eta}(\mathbf{n})),$$

(c)
$$\phi_D(x,\mathbf{n},s) \equiv \sum_{\substack{|\mathbf{k}|\leq x \\ (\mathbf{k},\mathbf{n})=\mathbf{1}}} |\mathbf{k}|^s = \frac{\Delta\delta}{s+\delta} \cdot \frac{\varphi_\delta(\mathbf{n})}{|\mathbf{n}|^\delta} x^{s+\delta} + O(x^{s+\eta+\varepsilon}\sigma_{-\eta-\varepsilon}(\mathbf{n})),$$

where $\varphi_\delta(\mathbf{n}) \equiv \varphi_{D,\delta}(\mathbf{n})$.

If in addition A is a cross-convolution, then

(d)
$$\sum_{\substack{|\mathbf{k}|\leq x \\ (\mathbf{k},\mathbf{n})\in(P_A)}} |\mathbf{k}|^s = \frac{\Delta\delta}{s+\delta} \cdot \frac{\varphi_\delta(\mathbf{n}_{Q_A})}{|\mathbf{n}_{Q_A}|^\delta} x^{s+\delta} + O(x^{s+\eta+\varepsilon} r_A(\mathbf{n})),$$

where $r_A(\mathbf{n}) = 1$ for Q_A finite and $r_A(\mathbf{n}) = \sigma_{-\eta-\varepsilon}(\mathbf{n})$ for Q_A infinite.

Proof. (a) Applying Theorem 5.1 and Theorem 3.3/ v) we get

$$\phi_A(x,\mathbf{n},s) = \sum_{\mathbf{d}\in A(\mathbf{n})} \mu_A(\mathbf{d})|\mathbf{d}|^s \left(\frac{\Delta\delta}{s+\delta}(x/|\mathbf{d}|)^{s+\delta} + O\left((x/|\mathbf{d}|)^{s+\eta+\varepsilon}\right)\right) =$$

$$= \frac{\Delta\delta}{s+\delta} x^{s+\delta} \sum_{\mathbf{d}\in A(\mathbf{n})} \frac{\mu_A(\mathbf{d})}{|\mathbf{d}|^\delta} + O\left(x^{s+\eta+\varepsilon} \sum_{\mathbf{d}|\mathbf{n}} |\mathbf{d}|^{-\eta-\varepsilon} \right).$$

(b) Use (a) for $s = 0$, $x = |\mathbf{n}|$ and $\varepsilon = 0$.

(c) Yields at once from (a) applied for $A = D$.

(d) (cf. [13], Lemma 7 in case $G = \mathbf{N}$) We have $(\mathbf{k}, \mathbf{n}) \in (P_A)$ if and only if $(\mathbf{k}, \gamma(\mathbf{n}_{Q_A})) = \mathbf{1}$, where $\gamma(\mathbf{m})$ denotes the product of distinct prime factors of $\mathbf{m}$. Hence from (c) we obtain

$$\sum_{\substack{|\mathbf{k}|\le x \\ (\mathbf{k},\mathbf{n})\in(P_A)}} |\mathbf{k}|^s = \sum_{\substack{|\mathbf{k}|\le x \\ (\mathbf{k},\gamma(\mathbf{n}_{Q_A}))=\mathbf{1}}} |\mathbf{k}|^s = \frac{\Delta\delta}{s+\delta} \cdot \frac{\phi_\delta(\gamma(\mathbf{n}_{Q_A}))}{|\gamma(\mathbf{n}_{Q_A})|^\delta} + O(x^{s+\eta+\varepsilon}\sigma_{-\eta-\varepsilon}(\gamma(\mathbf{n}_{Q_A}))).$$

Here $\varphi_\delta(\gamma(\mathbf{n}_{Q_A}))/|\gamma(\mathbf{n}_{Q_A})|^\delta = \varphi(\mathbf{n}_{Q_A})/|\mathbf{n}_{Q_A}|^\delta$ and if Q_A is finite, then $\sigma_{-\eta-\varepsilon}(\gamma(\mathbf{n}_{Q_A})) \le \tau(\prod_{p\in Q_A} p) = C$, a constant, which completes the proof.

Lemma 6.1 *If G satisfies the basic axiom and $s > 0$, then*

$$\sum_{|\mathbf{n}|\le x} \frac{\sigma_{-\eta}(\mathbf{n})}{|\mathbf{n}|^{s+\eta}} = \begin{cases} O(1), & \text{if } s+\eta > \delta, \\ O(\log^2 x), & \text{if } s = \delta, \eta = 0, \\ O(\log x), & \text{if } s+\eta = \delta, \eta > 0, \\ O(x^{\delta-s}\log x), & \text{if } s < \delta, \eta = 0, \\ O(x^{\delta-s-\eta}), & \text{if } s+\eta < \delta, \eta > 0, \end{cases}$$

Proof. We have

$$T(x) \equiv \sum_{|\mathbf{n}|\le x} \frac{\sigma_{-\eta}(\mathbf{n})}{|\mathbf{n}|^{s+\eta}} = \sum_{|\mathbf{de}|\le x} \frac{1}{|\mathbf{de}|^{s+\eta}} \cdot \frac{1}{|\mathbf{d}|^\eta} = \sum_{|\mathbf{d}|\le x} \frac{1}{|\mathbf{d}|^{s+2\eta}} \sum_{|\mathbf{e}|\le \frac{x}{|\mathbf{d}|}} \frac{1}{|\mathbf{e}|^{s+\eta}}.$$

For $s + \eta > \delta$ this is

$$T(x) = \sum_{|\mathbf{d}|\le x} \frac{1}{|\mathbf{d}|^{s+2\eta}} O(1) = O\left(\sum_{|\mathbf{d}|\le x} \frac{1}{|\mathbf{d}|^{s+2\eta}} \right) = O(1)$$

by Theorem 3.3/ i).

For $s + \eta = \delta$ we get

$$T(x) = \sum_{|\mathbf{d}|\le x} \frac{1}{|\mathbf{d}|^{\delta+\eta}} \sum_{|\mathbf{e}|\le \frac{x}{|\mathbf{d}|}} \frac{1}{|\mathbf{e}|^\delta} = \sum_{|\mathbf{d}|\le x} \frac{1}{|\mathbf{d}|^{\delta+\eta}} O\left(\log \frac{x}{|\mathbf{d}|}\right) = O\left(\log x \sum_{|\mathbf{d}|\le x} \frac{1}{|\mathbf{d}|^{\delta+\eta}} \right)$$

$$= \begin{cases} O(\log^2 x), & \text{if } \eta = 0, \\ O(\log x), & \text{if } \eta > 0, \end{cases}$$

by Theorem 3.3/ ii) and i).

Finally, for $s + \eta < \delta$ we obtain

$$T(x) = \sum_{|\mathbf{d}|\le x} \frac{1}{|\mathbf{d}|^{s+2\eta}} O\left(\left(\frac{x}{|\mathbf{d}|}\right)^{\delta-s-\eta} \right) = O\left(x^{\delta-s-\eta} \sum_{|\mathbf{d}|\le x} \frac{1}{|\mathbf{d}|^{\delta+\eta}} \right)$$

$$= \begin{cases} O(x^{\delta-s}\log x), & \text{if } \eta = 0, \\ O(x^{\delta-s-\eta}), & \text{if } \eta > 0, \end{cases}$$

by Theorem 3.3/ iv), ii) and i).

Lemma 6.2 *If G satisfies the basic axiom with $\eta = 0$ and $0 < \varepsilon < \delta - s$, then*

$$\sum_{|\mathbf{n}|\le x} \frac{\sigma_{-\varepsilon}(\mathbf{n})}{|\mathbf{n}|^{s+\varepsilon}} = O(x^{\delta-s-\varepsilon}).$$

Proof. Similarly to the proof of Lemma 6.1.,

$$\sum_{|\mathbf{n}|\le x} \frac{\sigma_{-\varepsilon}(\mathbf{n})}{|\mathbf{n}|^{s+\varepsilon}} = \sum_{|\mathbf{d}|\le x} \frac{1}{|\mathbf{d}|^{s+2\varepsilon}} \sum_{|\mathbf{e}|\le \frac{x}{|\mathbf{d}|}} \frac{1}{|\mathbf{e}|^{s+\varepsilon}}$$

$$= \sum_{|\mathbf{d}|\le x} \frac{1}{|\mathbf{d}|^{s+2\varepsilon}} O\left(\left(\frac{x}{|\mathbf{d}|}\right)^{\delta-s-\varepsilon}\right) = O\left(x^{\delta-s-\varepsilon} \sum_{|\mathbf{d}|\le x} \frac{1}{|\mathbf{d}|^{\delta+\varepsilon}}\right) = O(x^{\delta-s-\varepsilon}),$$

using the estimates of Proposition 3.3.

Theorem 6.2 *If A is a cross-convolution, g is a bounded arithmetical function, $s > 0$ and $f = g *_A E_s$, where $E_s(\mathbf{n}) = |\mathbf{n}|^s$ for every $n \in \mathbf{N}$, then*

$$\sum_{|\mathbf{n}|\le x} f(\mathbf{n}) = \frac{\Delta\delta}{s+\delta} C_A(g,s) x^{s+\delta} + O(R(x)),$$

where

$$C_A(g,s) = \sum_{\mathbf{n}\in G} \frac{g(\mathbf{n})\varphi_\delta(\mathbf{n}_{Q_A})}{|\mathbf{n}|^{s+\delta}|\mathbf{n}_{Q_A}|^\delta}$$

and $R(x) = x^{s+\eta}$ ($s > \delta - \eta$), $x^\delta \log^2 x$ ($s = \delta, \eta = 0$ and Q_A is infinite), $x^\delta \log x$ ($s = \delta - \eta, \eta > 0$ and Q_A is infinite or $s = \delta - \eta$ and Q_A is finite), x^δ ($s < \delta - \eta$).

Proof. We apply Theorem 6.1/d) valid for $s > -\eta$ and $0 \le \varepsilon < \delta - \eta$:

$$\sum_{|\mathbf{n}|\le x} f(\mathbf{n}) = \sum_{|\mathbf{d}|\le x} g(\mathbf{d}) \sum_{\substack{|\mathbf{e}|\le \frac{x}{|\mathbf{d}|} \\ (\mathbf{e},\mathbf{d})\in(P_A)}} |\mathbf{e}|^s =$$

$$= \sum_{|\mathbf{d}|\le x} g(\mathbf{d}) \left(\frac{\Delta\delta}{s+\delta} \cdot \frac{\varphi_\delta(\mathbf{d}_{Q_A})}{|\mathbf{d}_{Q_A}|^\delta} \left(\frac{x}{|\mathbf{d}|}\right)^{s+\delta} + O\left(\left(\frac{x}{|\mathbf{d}|}\right)^{s+\eta+\varepsilon} r_A(\mathbf{d})\right)\right)$$

$$= \frac{\Delta\delta}{s+\delta} x^{s+\delta} \sum_{|\mathbf{d}|\le x} \frac{g(\mathbf{d})\varphi_\delta(\mathbf{d}_{Q_A})}{|\mathbf{d}|^{s+\delta}|\mathbf{d}_{Q_A}|^\delta} + O\left(x^{s+\eta+\varepsilon} \sum_{|\mathbf{d}|\le x} \frac{r_A(\mathbf{d})}{|\mathbf{d}|^{s+\eta+\varepsilon}}\right)$$

$$= \frac{\Delta\delta}{s+\delta} x^{s+\delta} \sum_{\mathbf{d}\in G} \frac{g(\mathbf{d})\varphi_\delta(\mathbf{d}_{Q_A})}{|\mathbf{d}|^{s+\delta}|\mathbf{d}_{Q_A}|^\delta} + O\left(x^{s+\delta} \sum_{|\mathbf{d}|>x} \frac{1}{|\mathbf{d}|^{s+\delta}}\right) + O\left(x^{s+\eta+\varepsilon} \sum_{|\mathbf{d}|\le x} \frac{r_A(\mathbf{d})}{|\mathbf{d}|^{s+\eta+\varepsilon}}\right).$$

Here the first O-term is $O(x^{s+\delta}x^{-s-\delta+\delta}) = O(x^\delta)$ by Theorem 3.3/i). The second O-term is for Q_A finite and choosing $\varepsilon = 0$,

$$O\left(x^{s+\eta} \sum_{|\mathbf{d}|\le x} \frac{1}{|\mathbf{d}|^{s+\eta}}\right) = \begin{cases} O(x^{s+\eta}), & \text{if } s+\eta > \delta, \\ O(x^\delta \log x), & \text{if } s+\eta = \delta, \\ O(x^{s+\eta}x^{\delta-s-\eta}) = O(x^\delta), & \text{if } s+\eta < \delta, \end{cases}$$

applying Theorem 3.3.

For Q_A infinite the second O-term is

$$O\left(x^{s+\eta+\varepsilon}\sum_{|\mathbf{d}|\le x}\frac{\sigma_{-\eta-\varepsilon}(\mathbf{d})}{|\mathbf{d}|^{s+\eta+\varepsilon}}\right).$$

For $\varepsilon=0$ we get by Lemma 6.1

$$O\left(x^{s+\eta}\sum_{|\mathbf{d}|\le x}\frac{\sigma_{-\eta}(\mathbf{d})}{|\mathbf{d}|^{s+\eta}}\right)=\begin{cases}O(x^{s+\eta}), & \text{if } s+\eta>\delta,\\ O(x^{\delta}\log^2 x), & \text{if } s=\delta, \eta=0,\\ O(x^{\delta}\log x), & \text{if } s+\eta=\delta, \eta>0 \text{ or } s<\delta, \eta=0,\\ O(x^{\delta}), & \text{if } s+\eta<\delta, \eta>0.\end{cases}$$

In case Q_A infinite, $s<\delta, \eta=0$ this O-term can be improved choosing $0<\varepsilon<\delta-s$. We have from Lemma 6.2

$$O\left(x^{s+\varepsilon}\sum_{|\mathbf{d}|\le x}\frac{\sigma_{-\varepsilon}(\mathbf{d})}{|\mathbf{d}|^{s+\varepsilon}}\right)=O(x^{s+\varepsilon}x^{\delta-s-\varepsilon})=O(x^{\delta}).$$

Theorem 6.3 *If A is a cross-convolution and $s>0$, then*

(*i*) $$\sum_{|\mathbf{n}|\le x}\sigma_{A,s}(\mathbf{n})=\frac{\Delta\delta}{s+\delta}\cdot\frac{\zeta(s+\delta)}{\zeta_{Q_A}(s+2\delta)}x^{s+\delta}+O(R(x)),$$

(*ii*) $$\sum_{|\mathbf{n}|\le x}\varphi_{A,s}(\mathbf{n})=\frac{\Delta\delta}{s+\delta}\cdot\frac{1}{\zeta_{P_A}(s+\delta)}\prod_{p\in Q_A}\left(1-\frac{|\mathbf{p}|^{\delta}-1}{|\mathbf{p}|^{\delta}(|\mathbf{p}|^{s+\delta}-1)}\right)x^{s+\delta}+O(R(x)),$$

where $R(x)$ is given in Theorem 6.2,

(*iii*) $$\sum_{|\mathbf{n}|\le x}\sigma_{A,\delta}(\mathbf{n})=\frac{\Delta}{2}\cdot\frac{\zeta(2\delta)}{\zeta_{Q_A}(3\delta)}x^{2\delta}+O(S(x)),$$

(*iv*) $$\sum_{|\mathbf{n}|\le x}\varphi_{A,\delta}(\mathbf{n})=\frac{\Delta}{2}\cdot\frac{1}{\zeta_{P_A}(2\delta)}\prod_{p\in Q_A}\left(1-\frac{1}{|\mathbf{p}|^{\delta}(|\mathbf{p}|^{\delta}+1)}\right)x^{2\delta}+O(S(x)),$$

(*v*) $$\sum_{|\mathbf{n}|\le x}\phi_A(\mathbf{n})=\frac{\Delta^2}{2}\cdot\frac{1}{\zeta_{P_A}(2\delta)}\prod_{p\in Q_A}\left(1-\frac{1}{|\mathbf{p}|^{\delta}(|\mathbf{p}|^{\delta}+1)}\right)x^{2\delta}+O(S(x)),$$

where $S(x)=x^{\delta+\eta}$ ($\eta>0$), $x^{\delta}\log^2 x$ ($\eta=0$ and Q_A is infinite), $x^{\delta}\log x$ ($\eta=0$ and Q_A is infinite).

Proof. Apply Theorem 6.1 for $g(\mathbf{n})=I(\mathbf{n})=1$ and $g(\mathbf{n})=\mu_A(\mathbf{n})$, respectively, where by the Euler product formula,

$$C_A(I,s)=\sum_{\mathbf{n}\in G}\frac{\varphi_{\delta}(\mathbf{n}_{Q_A})}{|\mathbf{n}|^{s+\delta}|\mathbf{n}_{Q_A}|^{\delta}}=\prod_{\mathbf{p}\in P_A}\sum_{a=0}^{\infty}\frac{1}{|\mathbf{p}|^{s+\delta}}\prod_{\mathbf{p}\in Q_A}\sum_{a=0}^{\infty}\frac{\varphi_{\delta}(\mathbf{p}^a)}{|\mathbf{p}|^{s+2\delta}}$$

$$=\zeta_{P_A}(s+\delta)\zeta_{Q_A}(s+\delta)\zeta_{Q_A}^{-1}(s+2\delta)=\zeta(s+\delta)\zeta_{Q_A}^{-1}(s+2\delta)$$

and we obtain $C_A(\mu_A,s)$ in a similar manner, see also [13], Lemma 9 and Lemma 10.

7 Asymptotic estimates for divisor functions defined on an arithmetical semigroup

In this section we consider the divisor function $\tau_A(\mathbf{n})$, where A is a cross-convolution. For $A = D$ we have the usual divisor function $\tau_D(\mathbf{n}) \equiv \tau(\mathbf{n})$ and it is shown in [8], p. 92, that

Theorem 7.1

$$\sum_{|\mathbf{n}|\le x} \tau(\mathbf{n}) = \Delta x^\delta(\Delta\delta \log x + 2\gamma_G - \Delta) + O(x^{\frac{\delta+\eta}{2}}).$$

For the function $\tau_U(\mathbf{n}) \equiv \tau^*(\mathbf{n})$ in [8], p. 107 is given an analogous estimate.
In fact we investigate the function

$$\tau_H(\mathbf{n}) = \sum_{\substack{\mathbf{de}=\mathbf{n}\\ (\mathbf{d},\mathbf{e})\in H}} 1,$$

where H is a subset of G. Define the Möbius function of H by

$$\mu_H(\mathbf{n}) = \sum_{\mathbf{de}=\mathbf{n}} \rho(\mathbf{d})\mu(\mathbf{e}).$$

Lemma 7.1 *If H is an arbitrary subset of G, then for every $\mathbf{n}\in G$,*

$$\tau_H(\mathbf{n}) = \sum_{\mathbf{d}^2\mathbf{e}=\mathbf{n}} \mu_H(\mathbf{d})\tau(\mathbf{e})$$

Proof. Let ρ_H denote the characteristic function of H. We have

$$\tau_H(\mathbf{n}) = \sum_{\mathbf{de}=\mathbf{n}} \rho_H((\mathbf{d},\mathbf{e})) = \sum_{\mathbf{de}=\mathbf{n}}\sum_{\mathbf{a}|(\mathbf{d},\mathbf{e})} \mu_H(\mathbf{a}) = \sum_{a^2\mathbf{xy}=\mathbf{n}} \mu_H(\mathbf{a})$$

$$= \sum_{\mathbf{a}^2\mathbf{b}=\mathbf{n}} \mu_H(\mathbf{a}) \sum_{\mathbf{xy}=\mathbf{n}} 1 = \sum_{\mathbf{a}^2\mathbf{b}=\mathbf{n}} \mu_H(\mathbf{a})\tau(\mathbf{b}),$$

where $\mathbf{d} = \mathbf{ax}, \mathbf{e} = \mathbf{ay}$.

We will suppose that H is a multiplicative, i. e. $\mathbf{1}\in H$ and ρ_H is multiplicative. Note that if H is multiplicative, then $\mu_H(\mathbf{n})$ is also multiplicative and $\mu_H(\mathbf{n}) \in \{-1, 0, 1\}$ for every $\mathbf{n}\in G$.

Theorem 7.2 *If $H \subseteq G$ is multiplicative, then*

$$\sum_{|\mathbf{n}|\le x} \tau_H(\mathbf{n}) = \Delta x^\delta((\Delta\delta\log x + 2\gamma_G - \Delta)S_1 - 2\Delta\delta S_2) + O(W(x)),$$

where $W(x) = x^{\frac{\delta+\eta}{2}}$ for $\eta > 0$ and $W(x) = x^{\delta/2}\log x$ for $\eta = 0$ and

Proof. Using Lemma 7.1 and Theorem 7.1 we get

$$\sum_{|\mathbf{n}|\le x} \tau_H(\mathbf{n}) = \sum_{|\mathbf{d}^2\mathbf{e}|\le x} \mu_H(\mathbf{d})\tau(\mathbf{e}) = \sum_{|\mathbf{d}|\le\sqrt{x}} \mu_H(\mathbf{d}) \sum_{e\le\frac{x}{|\mathbf{d}|^2}} \tau(\mathbf{e})$$

$$= \sum_{|\mathbf{d}|\le\sqrt{x}} \mu_H(\mathbf{d}) \left(\Delta\left(\frac{x}{|\mathbf{d}|^2}\right)^\delta \left(\Delta\delta\log\left(\frac{x}{|\mathbf{d}|^2}\right) + 2\gamma_G - \Delta\right) + O\left(\frac{x}{|\mathbf{d}|^2}\right)^{\frac{\delta+\eta}{2}}\right)$$

$$= \Delta x^{\delta} \left((\Delta\delta \log x + 2\gamma_G - \Delta) \sum_{|\mathbf{d}|\le\sqrt{x}} \frac{\mu_H(\mathbf{d})}{|\mathbf{d}|^{2\delta}} - 2\Delta\delta \sum_{|\mathbf{d}|\le\sqrt{x}} \frac{\mu_H(\mathbf{d})\log|\mathbf{d}|}{|\mathbf{d}|^{2\delta}} \right)$$

$$+O\left(x^{\frac{\delta+\eta}{2}} \sum_{|\mathbf{d}|\le\sqrt{x}} \frac{|\mu_H(\mathbf{d})|}{|\mathbf{d}|^{\delta+\eta}} \right)$$

$$= \Delta x^{\delta} \left((\Delta\delta \log x + 2\gamma_G - \Delta) \sum_{\mathbf{d}\in G} \frac{\mu_H(\mathbf{d})}{|\mathbf{d}|^{2\delta}} + O\left(\log x \sum_{|\mathbf{d}|>\sqrt{x}} \frac{1}{|\mathbf{d}|^{2\delta}} \right) \right.$$

$$\left. -2\Delta\delta \sum_{\mathbf{d}\in G} \frac{\mu_H(\mathbf{d})\log|\mathbf{d}|}{|\mathbf{d}|^{2\delta}} + O\left(\sum_{|\mathbf{d}|>\sqrt{x}} \frac{\log|\mathbf{d}|}{|\mathbf{d}|^{2\delta}} \right) \right)$$

$$+O\left(x^{\frac{\delta+\eta}{2}} \sum_{|\mathbf{d}|\le\sqrt{x}} \frac{1}{|\mathbf{d}|^{\delta+\eta}} \right).$$

Here the first O-term is $O(x^{-\delta/2}\log x)$ and the third O-term is $O(x^{\frac{\delta+\eta}{2}}$ by Theorem 3.3/i).

Furthermore, it can be shown that

$$\sum_{|\mathbf{n}|\le x} \log|\mathbf{n}| = \Delta x^{\delta}\log x + O(x^{\delta}),$$

cf. [8], p. 90, and by partial summation, see [8], p. 82., we get

$$\sum_{|\mathbf{n}|>x} \frac{\log|\mathbf{n}|}{|\mathbf{n}|^{2\delta}} = O\left(\frac{\log x}{x^{\delta}}\right),$$

hence the second O-term is also $O(x^{-\delta/2}\log x)$ and the proof is complete.

8 Maximal and minimal orders of certain functions

Theorem 8.1 *If A is an arbitrary regular convolution, then the minimal order of the function $\varphi_{A,\delta}(\mathbf{n})$ is*

$$\frac{e^{-\gamma}|\mathbf{n}|^{\delta}}{\Delta \log\log|\mathbf{n}|},$$

where $\gamma \equiv \gamma_{G_Z}$ is the classical Euler constant.

Proof. This statement follows from the facts that this is known in case $A = D$, see [8], Theorem 3.2 and that $\varphi_{A,\delta}(\mathbf{n}) \ge \varphi_{D,\delta}(\mathbf{n}) \equiv \varphi_{\delta}(\mathbf{n})$ with equality for $\mathbf{n}$ squarefree.

More precisely,

$$\varphi_{A,\delta}(\mathbf{n}) = |\mathbf{n}|^{\delta} \prod_{\substack{\mathbf{p}\in P \\ \mathbf{n}(\mathbf{p})\ge 1}} \left(1 - |\mathbf{p}|^{-t\delta}\right) \ge |\mathbf{n}|^{\delta} \prod_{\substack{\mathbf{p}\in P \\ \mathbf{n}(\mathbf{p})\ge 1}} \left(1 - |\mathbf{p}|^{-\delta}\right) = \varphi_{\delta}(\mathbf{n}),$$

with equlity if $\mathbf{n}(\mathbf{p}) = 1$ for every $\mathbf{p}$, i. e. if $\mathbf{n}$ is squarefree, where $t = t_A(\mathbf{p}^{\mathbf{n}(\mathbf{p})})$.

According to the proof of [8], Theorem 3.2 for

$$\mathbf{n} = \mathbf{n}(x) = \prod_{|\mathbf{p}| \leq x} \mathbf{p}$$

we have

$$\varphi_{A,\delta}(\mathbf{n}) = \varphi_\delta(\mathbf{n}) \sim \frac{e^{-\gamma}|\mathbf{n}|^\delta}{\Delta \log\log|\mathbf{n}|} \qquad \text{as} \quad x \to \infty.$$

On the other hand for an arbitary $\mathbf{m} \in G$,

$$\varphi_{A,\delta}(\mathbf{m}) \geq \varphi_\delta(\mathbf{m}) > (1-\varepsilon)\frac{e^{-\gamma}|\mathbf{m}|^\delta}{\Delta \log\log|\mathbf{m}|} \qquad \text{for} \quad |\mathbf{m}| \geq x_0(\varepsilon), \quad \text{say}.$$

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References

[1] A. Beurling, Analyse de la loi asymptotique de la distribution des nombres premiers generalises I., Acta Math., **68** (1937), 255-291.

[2] E. Cohen, Arithmetical functions associated with the unitary divisors of an integer, Math. Z., **74** (1960), 66-80.

[3] E. Cohen, The number of unitary divisors of an integer, Amer. Math. Monthly, **67** (1960), 879-880.

[4] P. Haukkanen, Classical arithmetical identities involving a generalization of Ramanujan's sum, Ann. Acad. Sci. Fenn. Ser. A. I. Math. Dissertationes, **68** (1988), 1-69.

[5] P. Haukkanen, On generalizations of some identities of Ramanujan, Bull. Soc. Math. Belg. – Tijdschr. Belg. Wisk. Gen., Ser. B, **43** (1991), 43-57.

[6] E. M. Horadam, Arithmetical functions associated with the unitary divisors of a generalised integer, Amer. Math. Monthly, **69** (1962), 196-199.

[7] E. M. Horadam, Unitary divisor functions of a generalised integer, Portugal. Math, **24** (1965), 131-143.

[8] J. Knopfmacher, Abstract analytic number theory, North – Holland, Amsterdam – New York, 1975.

[9] E. Krätzel, Lattice points, Kluwer, Dordrecht-Boston- London, 1988.

[10] P. J. McCarthy, Introduction to arithmetical functions, Springer – Verlag, 1986.

[11] W. Narkiewicz, On a class of arithmetical convolutions, Colloq. Math., **10** (1963), 81-94.

[12] V. Sita Ramaiah, Arithmetical sums in regular convolutions, J. Reine Angew. Math., **303/304** (1978), 265-283.

[13] L. Tóth, Asymptotic formulae concerning arithmetical functions defined by cross-convolutions, I. Divisor-sum functions and Euler-type functions, Publ. Math. Debrecen, **50** (1997), 159-176.

[14] L. Tóth, Asymptotic formulae concerning arithmetical functions defined by cross-convolutions, II. The divisor function, Studia Univ. Babes-Bolyai, Math., **42** (1997), 105-110.

[15] L. Tóth, Sum functions of certain generalized divisors, Ann. Univ. Sci. Budapest. Rolando Eötvös, Sect. Math. **41** (1998), 167-182.